A SYNOPSIS

OF

ELEMENTARY RESULTS

IN

PURE MATHEMATICS:

CONTAINING

PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,

WITH

ABRIDGED DEMONSTRATIONS.

Supplemented by an Index to the Papers on Pure Mathematics which are to be found in the principal Journals and Transactions of learned Societies, both English and Foreign, of the present century.

BY

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PREFACE TO PART I.

The work, of which the part now issued is a first instalment, has been compiled from notes made at various periods of the last fourteen years, and chiefly during the engagements of teaching. Many of the abbreviated methods and mnemonic rules are in the form in which I originally wrote them for my pupils.

The general object of the compilation is, as the title indicates, to present within a moderate compass the fundamental theorems, formulæ, and processes in the chief branches of pure and applied mathematics.

The work is intended, in the first place, to follow and supplement the use of the ordinary text-books, and it is arranged with the view of assisting the student in the task of revision of book-work. To this end I have, in many cases, merely indicated the salient points of a demonstration, or merely referred to the theorems by which the proposition is proved. I am convinced that it is more beneficial to the student to recall demonstrations with such aids, than to read and re-read them. Let them be read once, but recalled often. The difference in the effect upon the mind between reading a mathematical demonstration, and originating one wholly or

partly, is very great. It may be compared to the difference between the pleasure experienced, and interest aroused, when in the one case a traveller is passively conducted through the roads of a novel and unexplored country, and in the other case he discovers the roads for himself with the assistance of a map.

In the second place, I venture to hope that the work, when completed, may prove useful to advanced students as an *aide-mémoire* and book of reference. The boundary of mathematical science forms, year by year, an ever widening circle, and the advantage of having at hand some condensed statement of results becomes more and more evident.

To the original investigator occupied with abstruse researches in some one of the many branches of mathematics, a work which gathers together synoptically the leading propositions in all, may not therefore prove unacceptable. Abler hands than mine undoubtedly, might have undertaken the task of making such a digest; but abler hands might also, perhaps, be more usefully employed,—and with this reflection I have the less hesitation in commencing the work myself. The design which I have indicated is somewhat comprehensive, and in relation to it the present essay may be regarded as tentative. The degree of success which it may meet with, and the suggestions or criticisms which it may call forth, will doubtless have their effect on the subsequent portions of the work.

With respect to the abridgment of the demonstrations, I may remark, that while some diffuseness of explanation is not only allowable but very desirable in an initiatory treatise, conciseness is one of the chief requirements in a work intended

for the purposes of revision and reference only. In order, however, not to sacrifice clearness to conciseness, much more labour has been expended upon this part of the subject-matter of the book than will at first sight be at all evident. The only palpable result being a compression of the text, the result is so far a negative one. The amount of compression attained is illustrated in the last section of the present part, in which more than the number of propositions usually given in treatises on Geometrical Conics are contained, together with the figures and demonstrations, in the space of twenty-four pages.

The foregoing remarks have a general application to the work as a whole. With the view, however, of making the earlier sections more acceptable to beginners, it will be found that, in those sections, important principles have sometimes been more fully elucidated and more illustrated by examples, than the plan of the work would admit of in subsequent divisions.

A feature to which attention may be directed is the uniform system of reference adopted throughout all the sections. With the object of facilitating such reference, the articles have been numbered progressively from the commencement in large Clarendon figures; the breaks which will occasionally be found in these numbers having been purposely made, in order to leave room for the insertion of additional matter, if it should be required in a future edition, without disturbing the original numbers and references. With the same object, demonstrations and examples have been made subordinate to enunciations and formulæ, the former being printed in small, the latter in bold

type. By these aids, the interdependence of propositions is more readily shown, and it becomes easy to trace the connexion between theorems in different branches of mathematics, without the loss of time which would be incurred in turning to separate treatises on the subjects. The advantage thus gained will, however, become more apparent as the work proceeds.

The Algebra section was printed some years ago, and does not quite correspond with the succeeding ones in some of the particulars named above. Under the pressure of other occupations, this section moreover was not properly revised before going to press. On that account the table of errata will be found to apply almost exclusively to errors in that section; but I trust that the list is exhaustive. Great pains have been taken to secure the accuracy of the rest of the volume. Any intimation of errors will be gladly received.

I have now to acknowledge some of the sources from which the present part has been compiled. In the Algebra, Theory of Equations, and Trigonometry sections, I am largely indebted to Todhunter's well-known treatises, the accuracy and completeness of which it would be superfluous in me to dwell upon.

In the section entitled Elementary Geometry, I have added to simpler propositions a selection of theorems from Townsend's Modern Geometry and Salmon's Conic Sections.

In Geometrical Conics, the line of demonstration followed agrees, in the main, with that adopted in Drew's treatise on the subject. I am inclined to think that the method of that author cannot be much improved. It is true that some important properties of the ellipse, which are arrived at in

Drew's Conic Sections through certain intermediate propositions, can be deduced at once from the circle by the method of orthogonal projection. But the intermediate propositions cannot on that account be dispensed with, for they are of value in themselves. Moreover, the method of projection applied to the hyperbola is not so successful; because a property which has first to be proved true in the case of the equilateral hyperbola, might as will be proved at once for the general case. I have introduced the method of projection but sparingly, always giving preference to a demonstration which admits of being applied in the same identical form to tho ellipse and to the hyperbola. The remarkable analogy subsisting between the two curves is thus kept prominently before the reader.

The account of the C. G. S. system of units given in the preliminary section, has been compiled from a valuable contribution on the subject by Professor Everett, of Belfast, published by the Physical Society of London.* This abstract, and the tables of physical constants, might perhaps have found a more appropriate place in an after part of the work. I have, however, introduced them at the commencement, from a sense of the great importance of the reform in the selection of units of measurement which is embodied in the C. G. S. system, and from a belief that the student cannot be too early familiarized with the same.

The Factor Table which follows is, to its limited extent, a reprint of Burckhardt's "*Tables des diviseurs*," published in

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* "Illustrations of the Centimetre-Gramme-Second System of Units." London: Taylor and Francis. 1875.

1814–17, which give the least divisors of all numbers from 1 to 3,036,000. In a certain sense, it may be said that this is the only sort of purely mathematical table which is absolutely indispensable, because the information which it gives cannot be supplied by any process of direct calculation. The logarithm of a number, for instance, may be computed by a formula. Not so its prime factors. These can only be arrived at through the tentative process of successive divisions by the prime numbers, an operation of a most deterrent kind when the subject of it is a high integer.

A table similar to and in continuation of Burckhardt's has recently been constructed for the fourth million by J. W. L. Glaisher, F.R.S., who I believe is also now engaged in completing the fifth and sixth millions. The factors for the seventh, eighth, and ninth millions were calculated previously by Dase and Rosenberg, and published in 1862–65, and the tenth million is said to exist in manuscript. The history of the formation of these tables is both instructive and interesting.*

As, however, such tables are necessarily expensive to purchase, and not very accessible in any other way to the majority of persons, it seemed to me that a small portion of them would form a useful accompaniment to the present volume. I have, accordingly, introduced the first eleven pages of Burckhardt's tables, which give the least factors of the first 100,000 integers nearly. Each double page of the table here printed is

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* See "*Factor Table for the Fourth Million.*" By James Glaisher, F.R.S. London: Taylor and Francis. 1880. Also *Camb. Phil. Soc. Proc.*, Vol. III., Pt. IV., and *Nature*, No. 542, p. 462.

an exact reproduction, in all but the type, of a single quarto page of Burckhardt's great work.

It may be noticed here that Prof. Lebesque constructed a table to about this extent, on the plan of omitting the multiples of seven, and thus reducing the size of the table by about one-sixth.* But a small calculation is required in using the table which counterbalances the advantage so gained.

The values of the Gamma-Function, pages 30 and 31, have been taken from Legendre's table in his "*Exercices de Calcul Intégral*," Tome I. The table belongs to Part II. of this Volume, but it is placed here for the convenience of having all the numerical tables of Volume I. in the same section.

In addition to the authors already named, the following treatises have been consulted—Algebras, by Wood, Bourdon, and Lefebure de Fourcy; Snowball's Trigonometry; Salmon's Higher Algebra; the Geometrical Exercises in Potts's Euclid; and Geometrical Conics by Taylor, Jackson, and Renshaw.

Articles 260, 431, 569, and very nearly all the examples, are original. The latter have been framed with great care, in order that they might illustrate the propositions as completely as possible.

G. S. C.

HADLEY, MIDDLESEX;
May 23, 1880.

* "Tables diverses pour la décomposition des nombres en leurs facteurs premiers." Par V. A. Lebesque. Paris. 1864.

ERRATA.

Art.	13,		*for*	$-a^2b^2$	*read*	$+a^2b^2$.
,,	56,	Line 1,	,,	3	,,	$\frac{3}{2}$.
,,	66,	,, 5,	,,	x	,,	x^2.
,,	90,	,, 4,	,,	numerators 1, 1, 1	,,	1, a, a^2.
,,	99,	,, 1,	,,	denominator $r-1$	,,	$n-1$.
,,	107,	,, 1,	,,	taken	,,	taken m at a time.
,,	108,	,, 2,	,,	(196)	,,	(360).
,,	131,	,, 1, 2,	,,	5	,,	6.
,,	,,	,, 5,	,,	$(-1)^5$	,,	$(-1)^{23}$.
,,	133,	,, 3, 6, 7,	,,	$6x$	,,	$3x$.
,,	,,	,, 8,	,,	4	,,	34.
,,	,,	,, 9,	,,	204, 459	,,	102, 306.
,,	,,	,, 10,	,,	459	,,	$9n$.
,,	138,	,, 4,	,,	$\frac{10.9.8}{1.2.3}$	,,	7.8.9.10.
,,	140,		,,	$(q+1)^k$	,,	$(q+1)^{(k)}$ Notation of (96).
,,	182,	,, 5,	,,	u_{n-1} in numerator	,,	u^2_{n-1}.
,,	191,	,, 4,	,,	(163)	,,	(164).
,,	220,	,, 6,	,,	$(x+y+z)^2$	,,	$2(x+y+z)^2$.
,,	221,	,, 4,	,,	(1)	,,	square of (1).
,,	237,	,, 11,	,,	$x^2=1$	,,	$x^2=-1$.
,,	238,	,, 5,	,,	(x^2-4x+8) on left side	,,	$(x^2-4x+8)^3$.
,,	239,	,, 11,		(234)		*Dele.*
,,	248,	,, 4,	,,	(29)	,,	(28).
,,	267,	,, 4,	,,	(267)	,,	(266).
,,	274,	,, 8,	,,	$\lfloor\underline{11}$	,,	$2\lfloor\underline{11}$.
,,	276,	,, 13,	,,	$p+2$	,,	$p+1$.
,,	,,	,, 14,	,,	$(p-1)$	,,	$\lfloor\underline{p-1}$.
,,	283,	,, 3,	,,	$x=1$	,,	$a=b$.
,,	288,	,, 7,	,,	$n-1$	,,	$n+1$.
,,	289,	,, 4,	,,	$H(r, n-1)$	,,	$H(n, r-1)$.
,,	290,	,, 2,	,,	$H(r+1, n-1)$	,,	$H(n, r)$.
,,	325,	,, 17,	,,	P_2	,,	P.
,,	,,		,,	$P_1P_2P_3$, last line but one	,,	$Q_1Q_2Q_3$.
,,	333,	,, 3,	,,	$\left(\frac{a+b}{2}\right)$	,,	$\left(\frac{a+b}{2}\right)^m$.
,,	361,	,, 7,	,,	3528	,,	10281.
,,	481,	,, 6,	,,	$n-3$	,,	$n-1$.
,,	514,	,, 4,		applying Descartes' rule		*Dele.*
,,	517,	,, 3,	,,	a^3	,,	x^3.
,,	544,	,, 1,				Transpose F and f.
,,	551,	,, 1,	,,	B_1	,,	B.
,,	,,	,, 9,	,,	$\alpha-n$	,,	$\alpha-\kappa$.
,,	704,		,,	(11, 12)	,,	(9, 10, 1).
,,	729,		,,	(940)	,,	(960).

Article 112 should be as follows:—

$$\frac{1}{1+2\sqrt{3}-\sqrt{2}} = \frac{1+2\sqrt{3}+\sqrt{2}}{(1+2\sqrt{3})^2-2} = \frac{1+2\sqrt{3}+\sqrt{2}}{11+4\sqrt{3}} = \frac{(1+2\sqrt{3}+\sqrt{2})(11-4\sqrt{3})}{73}.$$

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SECTION I.—MATHEMATICAL TABLES.

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INDEX TO PROPOSITIONS OF EUCLID

REFERRED TO IN THIS WORK.

The references to Euclid are made in Roman and Arabic numerals; *e.g.* (VI. 19).

BOOK I.

I. 4.—Triangles are equal and similar if two sides and the included angle of each are equal each to each.

I. 5.—The angles at the base of an isosceles triangle are equal.

I. 6.—The converse of 5.

I. 8.—Triangles are equal and similar if the three sides of each are equal each to each.

I. 16.—The exterior angle of a triangle is greater than the interior and opposite.

I. 20.—Two sides of a triangle are greater than the third.

I. 26.—Triangles are equal and similar if two angles and one corresponding side of each are equal each to each.

I. 27.—Two straight lines are parallel if they make equal alternate angles with a third line.

I. 29.—The converse of 27.

I. 32.—The exterior angle of a triangle is equal to the two interior and opposite; and the three angles of a triangle are equal to two right angles.

Cor. 1.—The interior angles of a polygon of n sides $= (n-2)\,\pi$.

Cor. 2.—The exterior angles $= 2\pi$.

I. 35 to 38.—Parallelograms or triangles upon the same or equal bases and between the same parallels are equal.

I. 43.—The complements of the parallelograms about the diameter of a parallelogram are equal.

I. 47.—The square on the hypotenuse of a right-angled triangle is equal to the squares on the other sides.

I. 48.—The converse of 47.

BOOK II.

II. 4.—If a, b are the two parts of a right line, $(a+b)^2 = a^2+2ab+b^2$.

If a right line be bisected, and also divided, internally or externally, into two unequal segments, then—

II. 5 and 6.—The rectangle of the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section; or $(a+b)(a-b) = a^2-b^2$.

II. 9 and 10.—The squares on the same unequal segments are together double the squares on the other parts; or

$$(a+b)^2+(a-b)^2 = 2a^2+2b^2.$$

II. 11.—To divide a right line into two parts so that the rectangle of the whole line and one part may be equal to the square on the other part.

II. 12 and 13.—The square on the base of a triangle is equal to the sum of the squares on the two sides *plus* or *minus* (as the vertical angle is *obtuse* or *acute*), twice the rectangle under either of those sides, and the projection of the other upon it; or $a^2 = b^2+c^2-2bc\cos A$ (702).

BOOK III.

III. 3.—If a diameter of a circle bisects a chord, it is perpendicular to it: and conversely.

III. 20.—The angle at the centre of a circle is twice the angle at the circumference on the same arc.

III. 21.—Angles in the same segment of a circle are equal.

III. 22.—The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

III. 31.—The angle in a semicircle is a right angle.

III. 32.—The angle between a tangent and a chord from the point of contact is equal to the angle in the alternate segment.

III. 33 and 34.—To *describe* or to *cut off* a segment of a circle which shall contain a given angle.

III. 35 and 36.—The rectangle of the segments of any chord of a circle drawn through an *internal* or *external* point is equal to the square of the semi-chord perpendicular to the diameter through the internal point, or to the square of the tangent from the external point.

III. 37.—The converse of 36. If the rectangle be equal to the square, the line which meets the circle touches it.

BOOK IV.

IV. 2.—To inscribe a triangle of given form in a circle.
IV. 3.—To describe the same about a circle.
IV. 4.—To inscribe a circle in a triangle.
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IV. 10.—To construct two-fifths of a right angle.
IV. 11.—To construct a regular pentagon.

BOOK VI.

VI. 1.—Triangles and parallelograms of the same altitude are proportional to their bases.

VI. 2.—A right line parallel to the side of a triangle cuts the other sides proportionally; and conversely.

VI. 3 and A.—The bisector of the *interior* or *exterior* vertical angle of a triangle divides the base into segments proportional to the sides.

VI. 4.—Equiangular triangles have their sides proportional homologously.

VI. 5.—The converse of 4.

VI. 6.—Two triangles are equiangular if they have two angles equal, and the sides about them proportional.

VI. 7.—Two triangles are equiangular if they have two angles equal and the sides about two *other* angles proportional, provided that the third angles are both greater than, both less than, or both equal to a right angle.

VI. 8.—A right-angled triangle is divided by the perpendicular from the right angle upon the hypotenuse into triangles similar to itself.

VI. 14 and 15.—Equal *parallelograms*, or *triangles* which have two angles equal, have the sides about those angles reciprocally proportional; and conversely, if the sides are in this proportion, the figures are equal.

VI. 19.—Similar triangles are in the duplicate ratio of their homologous sides.

VI. 20.—Likewise similar polygons.

VI. 23.—Equiangular parallelograms are in the ratio compounded of the ratios of their sides.

VI. B.—The rectangle of the sides of a triangle is equal to the square of the bisector of the vertical angle *plus* the rectangle of the segments of the base.

BOOK XI.

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PART II.

SECTION VIII.—DIFFERENTIAL CALCULUS.

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SECTION XI.—DIFFERENTIAL EQUATIONS.

No. of Article.

SECTION XII.—CALCULUS OF FINITE DIFFERENCES.

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ANALYTICAL CONICS IN CARTESIAN COORDINATES.

ANALYTICAL CONICS IN TRILINEAR COORDINATES.

THEORY OF PLANE CURVES.

SECTION XIV.—SOLID COORDINATE GEOMETRY.

PREFACE TO PART II.

Apologies for the non-completion of this volume at an earlier period are due to friends and enquirers. The labour involved in its production, and the pressure of other duties, must form the author's excuse.

In the compilation of Sections VIII. to XIV., the following works have been made use of:—

Treatises on the Differential and Integral Calculus, by Bertrand, Hymer, Todhunter, Williamson, and Gregory's Examples on the same subjects; Salmon's Lessons on Higher Algebra.

Treatises on the Calculus of Variations, by Jellett and Todhunter; Boole's Differential Equations and Supplement; Carmichael's Calculus of Operations; Boole's Calculus of Finite Differences, edited by Moulton.

Salmon's Conic Sections; Ferrers's Trilinear Coordinates; Kempe on Linkages (*Proc. of Roy. Soc.*, Vol. 23); Frost and Wolstenholme's Solid Geometry; Salmon's Geometry of Three Dimensions.

Wolstenholme's Problems.

The Index which concludes the work, and which, it is hoped, will supply a felt want, deals with 890 volumes of 32 serial publications: of these publications, thirteen belong to Great Britain, one to New South Wales, two to America, four to France, five to Germany, three to Italy, two to Russia, and two to Sweden.

As the volumes only date from the year 1800, the

important contributions of Euler to the "Transactions of the St. Petersburg Academy," in the last century, are excluded. It was, however, unnecessary to include them, because a very complete classified index to Euler's papers, as well as to those of David Bernoulli, Fuss, and others in the same Transactions, already exists.

The titles of this Index, and of the works of Euler therein referred to, are here appended, for the convenience of those who may wish to refer to the volumes.

Tableau général des publications de l'Académie Impériale de St. Pétersbourg depuis sa fondation. 1872. [B.M.C.:* *R.R.* 2050,*e.*]

I. Commentarii Academiæ Scientiarum Imperialis Petropolitanæ. 1726–1746; 14 vols. [B. M. C.: 431,*f.*]

II. Novi Commentarii A. S. I. P. 1747–75, 1750–77; 21 vols. [B. M. C.: 431,*f.*15–17, *g.*1–16, *h.*1, 2.]

III. Acta A. S. I. P. 1778–86; 12 vols. [B. M. C: 431,*h.*3–8; or *T.C.* 8,*a.*11.]

IV. Nova Acta A. S. I. P. 1787–1806; 15 vols. [B. M. C.: 431, *h.*9–15, *i.*1–8; or *T.C.* 8, *a.*23.]

V. Leonhardi Euler Opera minora collecta, vel Commentationes Arithmeticæ collectæ; 2 vols. 1849. [B. M. C.: 8534,*ee.*]

VI. Opera posthuma mathematica et physica; 2 vols. 1862. [B.M.C.: 8534,*f.*]

VII. Opuscula analytica; 1783–5; 2 vols. [B. M. C.: 50,*i.* 15.]

Analysis infinitorum. [B. M. C.: 529,*b.*11.]

G. S. C.

ENDSLEIGH GARDENS,
LONDON, N.W., 1886.

* British Museum Catalogue.

ERRATA CONTINUED FROM PAGE x.

(*Corrections which are important are marked with an asterisk.*)

	Page	1,	Line	7,	*for*	volume	*read*	weight.
			,,	10,	,,	gramme-million	,,	gramme-six.
*	,,	6,	,,	5,	,,	1·4971499	,,	·4971499.
*			,,	6,	,,	·6679358	,,	1·1447299.
*	Art.	123,	,,	2,	,,	$2\sqrt{5}$	,,	$2\sqrt{15}$.
*	,,	259,	,,	2,	,,	$\alpha^2+\beta^2$	,,	$(\alpha^2+\beta^2)^n$.
*	,,	276,	,,	6,	,,	$3n^2+n-1$	,,	$3n^2+3n-1$.
	,,	291,	,,	1,	,,	δ	,,	γ.
	,,	292,	,,	3 & 4,	,,	a	,,	α.
*	,,	322,	,,	9,	,,	45 and 13	,,	35 and 10.
*	,,	361,	,,	7,	,,	3528	,,	8584.
*	,,	459,	,,	3 & 9,	,,	-6	,,	-16.
	,,	470,	,,	1,	,,	x_m	,,	x .
*	,,	489,	,,	9,			,,	$-\frac{4q}{3}$.
*	,,	555,	,,	14,	,,	a number of rows	,,	two columns.
	,,	593,	,,	11 & 12,	,,	R and R	,,	R_1 and R_2.
	,,	604,	,,	2,	,,	one-sixtieth	,,	one-ninetieth.
	,,	713,	,,	2,	,,	II.	,,	III.
*	,,	897,	,,	5,	,,	$\cos\frac{1}{2}c$	,,	$\sin\frac{1}{2}c$.
	,,	922 ii.,			,,	b^2+2c^2	,,	$2b^2+c^2$.
	,,	949,	,,	last,	,,	D	,,	C.
*	,,	1076,	,,	2,		*dele* "The projections . . . are parallel."		
	,,	1158,	,,	last,	,,	1201	,,	1217.
	,,	1178,	,,	4,	,,	PS	,,	PS'.
	,,	1241,	,,	1,	,,	parallel	,,	conjugate.
	,,	1413,	,,	3,	,,	$-du\,dv$	,,	$+du\,dv$.
	,,	1491,	,,	3,	,,	$-$	,,	$=$
	,,	1849,	,,	1,	,,	$+$	,,	$-$
	,,	1903,	*footnote*,		,,	$\int(x)$	,,	$f(x)$.
	,,	1925,				*supply* dx.		
	,,	1954-6,				*supply* x.		
	,,	2030-2,				erroneous, because l in (1427) is necessarily an integer.		
	,,	2035,	,,	1,	,,	ax	,,	a.
*	,,	2140,	,,	1,		$\surd$ applies to the whole denominator.		
	,,	2136,	,,	last,	,,	2294	,,	2293.
	,,	2354,			,,	$-$	,,	$+$
*	,,	2392,			,,	x^p	,,	$x\ -1$.
*	,,	2465,			,,	$\frac{p}{2}$	,,	$\frac{\pi}{2}$.
	,,	3237,	,,	1,	,,	$(n-1)x$	,,	$(n-1)$.
	,,	3751,				*supply* u_x.		
*	,,	4678,				*dele* 2 in the second term.		
*	,,	4680,				*supply* the factor 4 on the left.		
*	,,	4692,	,,	5,		*dele* 2 in the second term.		
*	,,	4903,	,,	3,		*supply* the factor 4 on the left.		
	,,	5154,	,,	4,	,,	3155	,,	5155.
	,,	5330,	,,	2,	,,	m	,,	b.
						and refer to Fig. 129, Art. 5332 on the cardioid is wanting.		

MATHEMATICAL TABLES.

INTRODUCTION.

The Centimetre-Gramme-Second system of units.

NOTATION.—The decimal measures of length are the *kilometre*, *hectometre*, *decametre*, *metre*, *decimetre*, *centimetre*, *millimetre*. The same prefixes are used with the *litre* and *gramme* for measures of capacity and ~~volume~~. weight correcte

Also, 10^7 metres is denominated a *metre-seven;* 10^{-7} metres, a *seventh-metre;* 10^{15} grammes, a *gramme-fifteen;* and so on.

A gramme-~~million~~ six is also called a *megagramme;* and a millionth-gramme, a *microgramme;* and similarly with other measures.

DEFINITIONS.—The C. G. S. system of units refers all physical measurements to the *Centimetre* (cm.), the *Gramme* (gm.), and the *Second* (sec.) as the units of length, mass, and time.

The quadrant of a meridian is approximately a *metre-seven*. More exactly, one metre = 3·2808694 feet = 39·370432 inches.

The *Gramme* is the *Unit of mass*, and the *weight of a gramme* is the *Unit of weight*, being approximately the weight of a cubic centimetre of water; more exactly, 1 gm. = 15·432349 grs.

The *Litre* is a cubic decimetre: but *one cubic centimetre* is the C. G. S. *Unit of volume*.

1 litre = ·035317 cubic feet = ·2200967 gallons.

The *Dyne* (dn.) is the *Unit of force*, and is the force which, in one second generates in a gramme of matter a velocity of one centimetre per second.

The *Erg* is the *Unit of work and energy*, and is the work done by a dyne in the distance of one centimetre.

The absolute *Unit of atmospheric pressure* is one megadyne per square centimetre = 74·964 cm., or 29·514 in. of mercurial column at 0° at London, where g = 981·17 dynes.

Elasticity of Volume = k, is the pressure per unit area upon a body divided by the cubic dilatation.

Rigidity $= n$, is the shearing stress divided by the angle of the shear.

Young's Modulus $= M$, is the longitudinal stress divided by the elongation produced, $= 9nk \div (3k+n)$.

Tenacity is the tensile strength of the substance in dynes per square centimetre.

The *Gramme-degree* is the *Unit of heat*, and is the amount of heat required to raise by 1° C. the temperature of 1 gramme of water at or near 0°.

Thermal capacity of a body is the increment of heat divided by the increment of temperature. When the increments are small, this is the thermal capacity *at* the given temperature.

Specific heat is the thermal capacity of unit mass of the body at the given temperature.

The *Electrostatic unit* is the quantity of electricity which repels an equal quantity at the distance of 1 centimetre with the force of 1 dyne.

The *Electromagnetic unit* of quantity $= 3 \times 10^{10}$ *electrostatic units* approximately.

The *Unit of potential* is the potential of unit quantity at unit distance.

The *Ohm* is the common *electromagnetic unit* of resistance, and is approximately $= 10^9$ *C. G. S. units*.

The *Volt* is the *unit of electromotive force*, and is $= 10^8$ *C. G. S. units of potential*.

The *Weber* is the *unit of current*, being the current due to an electromotive force of 1 Volt, with a resistance of 1 Ohm. It is $= \frac{1}{10}$ *C. G. S. unit*.

Resistance of a Wire = Specific resistance × Length ÷ Section.

Physical constants and Formulæ.

In the latitude of London, $g = 32{\cdot}19084$ feet per second.
$= 981{\cdot}17$ centimetres per second.

In latitude λ, at a height h above the sea level,

$$g = (980{\cdot}6056 - 2{\cdot}5028 \cos 2\lambda - {\cdot}000003\,h) \text{ centimetres per second.}$$

Seconds pendulum $= (99{\cdot}3562 - {\cdot}2536 \cos 2\lambda - {\cdot}0000003\,h)$ centimetres.

THE EARTH.—Semi-polar axis, 20,854895 feet* $= 6{\cdot}35411 \times 10^8$ centims.
Mean semi-equatorial diameter, 20,926202 „ * $= 6{\cdot}37824 \times 10^8$ „
Quadrant of meridian, $39{\cdot}377786 \times 10^7$ inches* $= 1{\cdot}000196 \times 10^7$ metres.
Volume, 1·08279 cubic centimetre-nines.
Mass (with a density $5\frac{1}{2}$) = Six gramme-twenty-sevens nearly.

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* These dimensions are taken from Clarke's "Geodesy," 1880.

Velocity in orbit = 2933000 centims per sec. Obliquity, $23° 27' 16''$.*
Angular velocity of rotation = $1 \div 13713$.
Precession, $50''{\cdot}26$.* Progression of Apse, $11''{\cdot}25$. Eccentricity, $e = {\cdot}01679$.
Centrifugal force of rotation at the equator, 3·3912 dynes per gramme.
Force of attraction upon moon, ·2701. Force of sun's attraction, ·5839.
Ratio of g to centrifugal force of rotation, $g : r\omega^2 = 289$.
Sun's horizontal parallax, $8''{\cdot}7$ to $9''$.* Aberration, $20''{\cdot}11$ to $20''{\cdot}79$.*
Semi-diameter at earth's mean distance, $16' 1''{\cdot}82$.*
Approximate mean distance, 92,000000 miles, or 1·48 centimetre-thirteens.†
Tropical year, 365·242216 days, or 31,556927 seconds.
Sidereal year, 365·256374 ,, 31,558150 ,,
Anomalistic year, 365·259544 days. Sidereal day, 86164 seconds.

THE MOON.—Mass = Earth's mass $\times {\cdot}011364 = 6{\cdot}98\ 10^{25}$ grammes.
Horizontal parallax. From $53' 56''$ to $61' 24''$.*
Sidereal revolution, 27d. 7h. 43m. 11·46s. Lunar month, 29d. 12h. 44m. 2·87s.
Greatest distance from the earth, 251700 miles, or 4·05 centimetre-tens,
Least ,, ,, 225600 ,, 3·63 ,, ,,
Inclination of Orbit, $5° 9'$. Annual regression of Nodes, $19° 20'$.

RULE.—(*The Year* + 1) ÷ 19. *The remainder is the Golden Number.*
(*The Golden Number* − 1) × 11 ÷ 30. *The remainder is the Epact.*

GRAVITATION.— Attraction between masses m, m' at a distance l $= \dfrac{mm'}{l^2 \times 1{\cdot}543 \times 10^7}$ dynes.

The mass which at unit distance (1 cm.) attracts an equal mass with unit force (1 dn.) is $= \sqrt{(1{\cdot}543 \times 10^7)}$ gm. = 3928 gm.

WATER.—Density at 0° C., unity; at 4°, 1·000013 (Kupffer).
Volume elasticity at 15°, $2{\cdot}22 \times 10^{10}$.
Compression for 1 megadyne per sq. cm., $4{\cdot}51 \times 10^{-5}$ (Amaury and Descamps).
The heat required to raise the temperature of a mass of water from 0° to $t°$ is proportional to $t + {\cdot}00002t^2 + {\cdot}0000003t^3$ (Regnault).

GASES.—Expansion for 1° C., $\cdot 003665 = 1 \div 273$.

$$\frac{\text{Specific heat at constant pressure}}{\text{Specific heat at constant volume}} = 1{\cdot}408.$$

Density of dry air at 0° with Bar. at 76 cm. = ·0012932 gm. per cb. cm. (Regnault).
At unit pres. (a megadyne) Density = ·0012759.
Density at press. $p = p \times 1{\cdot}2759 \times 10^{-9}$.

Density of saturated steam at $t°$, with p taken from Table II., is approximately $= \dfrac{{\cdot}7936098p}{(1 + {\cdot}00366t)\,10^9}$.

SOUND.—Velocity = $\sqrt{(\textit{elasticity of medium} \div \textit{density})}$.
Velocity in dry air at $t° = 33240\sqrt{(1 + {\cdot}00366t)}$ centimetres per second.
Velocity in water at $0° = 143000$,, ,,

LIGHT.—Velocity in a medium of absolute refrangibility μ
$= 3{\cdot}004 \times 10^{10} \div \mu$ (Cornu).

If P be the pressure in dynes per sq. cm., and t the temperature,
$\mu - 1 = 2903 \times 10^{-13} P \div (1 + {\cdot}00366t)$ (Biot & Arago).

* These data are from the "Nautical Almanack" for 1883.
† Transit of Venus, 1874, "Astrom. Soc. Notices," Vols. 37, 38.

Table I.

Various Measures and their Equivalents in C. G. S. units.

Dimensions.

1 inch	= 2·5400 cm.
1 foot	= 30·4797 ,,
1 mile	= 160933 ,,
1 nautical do.	= 185230 ,,
1 sq. inch	= 6·4516 sq. cm.
1 sq. foot	= 929·01 ,,
1 sq. yard	= 8361·13 ,,
1 sq. mile	= $2·59 \times 10^{10}$,,
1 cb. inch	= 16·387 cb. cm.
1 cb. foot	= 28316 ,,
1 cb. yard	= 764535 ,,
1 gallon	= 4541 ,,
	= 277·274 cb. in. or the volume of 10 lbs. of water at 62° Fah., Bar. 30 in.

Mass.

1 grain	= ·06479895 gm.
1 ounce	= 28·3495 ,,
1 pound	= 453·5926 ,,
1 ton	= 1,016047 ,,
1 kilogramme	= 2·20462125 lbs.
1 pound Avoir.	= 7000 grains
1 pound Troy	= 5760 ,,

Velocity.

1 mile per hour	= 44·704 cm. per sec.
1 kilometre ,,	= 27·777 ,,

Pressure.

1 gm. per sq. cm.	= 981 dynes per sq. cm.
1 lb. per sq. foot	= 479 ,,
1 lb. per sq. in.	= 68971 ,,
76 centimetres of mercury at 0° C.	= 1,014,000 ,,

$$\frac{\text{lbs. per sq. in.}}{\text{gms. per sq. cm.}} = 70·307 = \frac{1}{·014223}$$

Force of Gravity.

upon 1 gramme	= 981 dynes
,, 1 grain	= 63·56777 ,,
,, 1 oz.	= $2·7811 \times 10^4$,,
,, 1 lb.	= $4·4497 \times 10^5$,,
,, 1 cwt.	= $4·9837 \times 10^7$,,
,, 1 ton	= $9·9674 \times 10^8$,,

Work ($g = 981$).

1 gramme-centimetre	= 981 ergs.
1 kilogram-metre	= 981×10^5 ,,
1 foot-grain	= $1·937 \times 10^3$,,
1 foot-pound	= $1·356 \times 10^7$,,
1 foot-ton	= $3·04 \times 10^{10}$,,
1 'horse power' p. sec.	= $7·46 \times 10^9$,,

Heat.

1 gramme-degree C.	= 42×10^6 ergs.
1 pound-degree	= 191×10^8 ,,
1 pound-degree Fah.	= 106×10^8 ,,

Table II.

Pressure of Aqueous Vapour in dynes per square centim.

Temp.	Pressure.	Temp.	Pressure.
−20°	1236	40°	73200
−15°	1866	50°	122600
−10°	2790	60°	198500
− 5°	4150	80°	472900
0°	6133	100°	1014000
5°	8710	120°	1988000
10°	12220	140°	3626000
15°	16930	160°	6210000
20°	23190	180°	10060000
25°	31400	200°	15600000
30°	42050		

Table III.

Values for the principal Lines of the Spectrum in air at 160° C. with Bar. 76 cm.

	Wave-length in centims.	No. of vibrations per second.
A	$7·604 \times 10^{-5}$	$3·950 \times 10^{14}$
B	6·867 ,,	4·373 ,,
C	6·56201 ,,	4·577 ,,
D (mean)	5·89212 ,,	5·097 ,,
E	5·26913 ,,	5·700 ,,
F	4·86072 ,,	6·179 ,,
G	4·30725 ,,	6·973 ,,
H_1	3·96801 ,,	7·569 ,,
H_2	3·93300 ,,	7·636 ,,

TABLE IV. *in C. G. S. units.*

	Density Water = 1.	Young's Modulus M.	Rigidity n.	Elasticity of volume k.	Tenacity.	Expansion of Volume per degree C.	Linear Expansion between 0 & 100 C.	Specific Heat between 0 & 100 C.	Relative Conductivity.	Rate of Conduction of Sound in cm. per sec.	Elect. Magn. Specific Resistance at 0° C.
Platinum	21	—	—	—	—	·000027	·000875	·0335	381	$2·69 \times 10^5$	9158
Gold	19·26	—	—	—	—	·000045	·001483	—	1000	1·74 „	2081
Mercury	13·596	—	—	$0·542 \times 10^{12}$	—	·000180	—	·0330	—	—	96190
Lead	11·35	$·059 \times 10^{12}$	—	—	$2·28 \times 10^8$	·000086	·002861	—	180	1·23 „	19850
Silver	10·47	—	—	—	—	·000061	·00196	·0557	973	2·61 „	1521
Copper	8·843	1·234 „	$4·47 \times 10^{11}$	1·684 „	41·4 „	·000054	·00175	0949	898	3·74 „	1615
Brass, drawn	8·471	1·075 „	3·66 „	—	33·8 „	·000054	·00193	—	357	3·56 „	—
Iron, cast	7·235	1·349 „	5·32 „	0·964 „	—	·000033	·001111	—	—	4·32 „	—
Iron, wrought	7·677	1·963 „	7·69 „	1·456 „	58·6 „	·000040	·001258	·1098	374	5·06 „	9827
Steel	7·849	2·139 „	8·19 „	1·841 „	79·3 „	·000037	001260	—	—	5·22 „	—
Tin, cast	7·29	—	—	—	3·17 „	·000063	·00227	—	304	—	13360
Zinc, cast	7·19	—	—	—	5·17 „	·000088	·00294	·0927	363	—	5690
Glass, flint	2·942	0·603 „	2·40 „	0·415 „	—	·000015	·00081	·1770	—	4·53 „	—

TABLE V.

	Greatest distance from Sun. Earth's mean distance = 1.	Least distance from Sun.	Sidereal Revolution in Days.	Inclination of Orbit to Ecliptic. °	′	″	Time of Rotation. h.	m.	s.	Diameter in Miles.	Mass.	Density.
Sun	—	—	—		—		600	0	0	888000	354936	0·25
Mercury	0·46669	0 30750	87·969	7	0	8	24	5	28	3000	0·118	2·01
Venus	0·72826	0·71840	224·701	3	23	31	23	21	21	7700	0·883	0·97
Earth	1·01678	0·98322	365·256		—		23	56	4	7926	1·000	1·00
Mars	1·66578	1·38160	686·980	1	51	5	24	37	22	4500	0 132	0·72
Jupiter	5·45378	4·95182	4332·585	1	18	40	9	55	26	92000	338·034	0·24
Saturn	10·07328	9·00442	10759·220	2	29	28	10	29	17	75000	101·064	0·13
Uranus	20·07612	18·28916	30686·821		46	30		—		36000	14·789	0·15
Neptune	30·29888	29·77506	60126·722	1	46	59		—		35000	24·648	0·27

TABLE VI.—*Functions of π and e.*

π = 3·1415926	π^{-1} = ·3183099	e = 2·71828183
π^2 = 9·8696044	π^{-2} = ·1013212	e^2 = 7·38905611
π^3 = 31·0062761	π^{-3} = ·0322515	e^{-1} = 0·3678794
$\sqrt{\pi}$ = 1·7724539	$200^g \div \pi$ = 63^g·6619772	e^{-2} = 0·1353353
$\log_{10} \pi$ = 1·4971499	$180° \div \pi$ = 57°·2957795	$\log_{10} e$ = 0·43429448
$\log_e \pi$ = ~~0·6679358~~ 1·1447298858	= 206264″·8	$\log_e 10$ = 2·30258509

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TABLE VII.

No.	Square root.	Cube root.
2	1·4142136	1·2599210
3	1·7320508	1·4422496
4	2·0000000	1·5874011
5	2·2360680	1·7099759
6	2·4494897	1·8171206
7	2·6457513	1·9129312
8	2·8284271	2·0000000
9	3·0000000	2·0800837
10	3·1622777	2·1544347
11	3·3166248	2·2239801
12	3·4641016	2·2894286
13	3·6055513	2·3513347
14	3·7416574	2·4101422
15	3·8729833	2·4662121
16	4·0000000	2·5198421
17	4·1231056	2·5712816
18	4·2426407	2·6207414
19	4·3588989	2·6684016
20	4·4721360	2·7144177
21	4·5825757	2·7589243
22	4·6904158	2·8020393
23	4·7958315	2·8438670
24	4·8989795	2·8844991
25	5·0000000	2·9240177
26	5·0990195	2·9624960
27	5·1961524	3·0000000
28	5·2915026	3·0365889
29	5·3851648	3·0723168
30	5·4772256	3·1072325

TABLE VIII.

N.	$\log_{10} N$.	$\log_e N$.
2	·3010300	·69314718
3	·4771213	1·09861229
5	·6989700	1·60943791
7	·8450980	1·94591015
11	1·0413927	2·39789527
13	1·1139434	2·56494936
17	1·2304489	2·83321334
19	1·2787536	2·94443898
23	1·3617278	3·13549422
29	1·4623980	3·36729583
31	1·4913617	3·43398720
37	1·5682017	3·61091791
41	1·6127839	3·71357207
43	1·6334685	3·76120012
47	1·6720979	3·85014760
53	1·7242759	3·97029191
59	1·7708520	4·07753744
61	1·7853298	4·11087386
67	1·8260748	4·20469262
71	1·8512583	4·26267988
73	1·8633229	4·29045944
79	1·8976271	4·36944785
83	1·9190781	4·41884061
89	1·9493900	4·48863637
97	1·9867717	4·57471098
101	2·0043214	4·61512052
103	2·0128372	4·63472899
107	2·0293838	4·67282883
109	2·0374265	4·69134788

NOTE.—The authorities for Table IV. are as follows:—Columns 2, 3, and 4 (Mercury excepted), Everett's experiments (Phil. Trans., 1867); g is here taken = 981·4. The densities in these cases are those of the specimens employed. Cols. 5 and 7, Rankine. Col. 6, Watt's Dict. of Chemistry. Col. 8, Dulong and Petit. Col. 10, Wertheim. Col. 11, Matthiessen.

Table V. is abridged from Loomis's Astronomy.

The values in Table III. are Angström's.

BURCKHARDT'S FACTOR TABLES.

For all numbers from 1 to 99000.

Explanation.—The tables give the least divisor of every number from 1 up to 99000: but numbers divisible by 2, 3, or 5 are not printed. All the digits of the number whose divisor is sought, excepting the units and tens, will be found in one of the three rows of larger figures. The two remaining digits will be found in the left-hand column. The least divisor will then be found in the column of the first named digits, and in the row of the units and tens.

If the number be prime, a cipher is printed in the place of its least divisor.

The numbers in the first left-hand column are not consecutive. Those are omitted which have 2, 3, or 5 for a divisor. Since $2^2.3.5^2 = 300$, it follows that this column of numbers will re-appear in the same order after each multiple of 300 is reached.

Mode of using the Tables.—If the number whose prime factors are required is divisible by 2 or 5, the fact is evident upon inspection, and the division must be effected. The quotient then becomes the number whose factors are required. If this number, being within the range of the tables, is yet not given, *it is divisible by* 3. Dividing by 3, we refer to the tables again for the new quotient and its least factor, and so on.

Examples.—Required the prime factors of 310155.

Dividing by 5, the quotient is 62031. This number is within the range of the tables. But it is not found printed. Therefore 3 is a divisor of it. Dividing by 3, the quotient is 20677. The table gives 23 for the least factor of 20677. Dividing by 23, the quotient is 899.

The table gives 29 for the least factor of 899. Dividing by 29, the quotient is 31, a prime number. Therefore $310155 = 3.5.23.29.31$.

Again, required the divisors of 92881. The table gives 293 for the least divisor. Dividing by it, the quotient is 317. Referring to the tables for 317, a cipher is found in the place of the least divisor, and this signifies that 317 is a prime number.

Therefore $92881 = 293 \times 317$, the product of two primes.

It may be remarked that, to have resolved 92881 into these factors without the aid of the tables by the method of Art. 360, would have involved fifty-nine fruitless trial divisions by prime numbers.

	00	03	06	09	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87
01	0	7	0	17	0	19	0	11	7	37	0	0	13	47	0	7	0	0	11	0	17	0	7	67	19	13	29	0	31	7
07	0	0	0	0	17	11	13	7	29	0	31	0	0	0	7	0	11	0	0	13	0	7	0	0	0	0	37	11	7	0
11	0	0	13	0	7	0	0	0	0	0	0	7	23	0	0	13	17	19	7	0	0	0	11	0	0	7	73	0	13	31
13	0	0	0	11	0	17	7	0	19	0	23	0	0	7	11	0	0	0	0	29	7	59	17	31	0	11	13	7	47	0
17	0	0	0	7	0	37	23	29	0	11	7	31	0	0	0	0	0	7	0	0	11	0	13	0	7	0	0	0	19	23
19	0	11	0	0	23	7	17	13	41	0	0	0	7	0	0	0	61	0	0	7	13	71	0	11	0	73	7	23	0	0
23	0	17	7	13	0	0	0	11	0	7	0	0	0	0	41	0	7	47	11	59	19	0	37	7	31	0	0	0	0	11
29	0	7	17	0	0	11	31	0	7	0	13	0	19	0	0	7	11	23	61	17	0	0	7	13	0	0	0	11	0	7
31	0	0	0	7	0	0	0	0	11	0	7	0	0	0	0	23	0	7	0	11	37	13	19	29	7	17	41	47	0	0
37	0	0	7	0	0	29	11	0	0	7	0	47	0	31	19	13	7	11	0	0	0	0	0	7	0	0	17	79	11	0
41	0	11	0	0	17	23	7	0	0	0	0	13	11	7	0	19	47	53	0	0	7	17	29	11	13	0	0	7	23	0
43	0	7	0	23	11	0	19	0	7	13	17	0	0	0	0	7	29	37	0	0	0	0	7	53	0	19	11	17	0	7
47	0	0	0	0	29	7	0	19	0	41	11	0	7	0	31	0	37	0	13	7	0	11	17	0	0	0	7	0	0	0
49	7	0	11	13	0	0	43	7	31	0	0	17	41	11	7	0	13	19	0	0	23	7	61	0	11	0	47	29	7	13
53	0	0	0	0	7	0	17	0	11	0	43	7	13	59	0	29	23	0	7	11	0	0	0	17	0	7	0	31	79	0
59	0	0	0	7	0	0	11	17	0	31	7	0	0	37	0	47	43	7	53	13	73	0	0	0	7	0	29	41	11	19
61	0	19	0	31	13	7	0	0	23	11	0	0	7	17	0	0	0	13	43	7	11	0	0	0	53	0	7	0	0	0
67	0	0	23	0	7	0	0	11	0	0	0	7	19	0	17	0	31	0	7	73	0	0	59	0	13	7	0	0	0	11
71	0	7	11	0	31	0	0	13	7	17	37	0	0	11	0	7	0	0	0	29	13	23	7	0	11	67	17	0	43	7
73	0	0	0	7	19	11	0	41	0	47	7	0	0	29	0	17	11	7	13	23	0	0	0	19	7	0	0	11	37	31
77	7	13	0	0	0	19	0	7	0	0	17	11	0	41	7	23	0	31	0	53	59	7	11	0	19	0	0	13	7	67
79	0	0	7	11	0	0	0	0	37	7	0	31	13	23	11	19	7	0	0	0	0	0	0	7	29	11	0	0	61	0
83	0	0	0	0	0	0	7	37	13	11	0	17	29	7	0	0	19	71	0	0	7	13	41	0	0	0	0	7	17	0
89	0	0	13	23	0	7	0	11	19	0	0	0	7	0	0	13	0	0	11	7	0	0	0	29	37	0	7	19	13	11
91	7	17	0	0	0	37	31	7	47	0	11	0	0	13	7	0	67	29	17	0	0	7	0	0	23	0	13	0	7	59
97	0	0	17	0	0	0	7	13	11	0	19	43	0	7	0	0	59	0	23	11	7	0	37	0	0	71	53	7	29	19

	01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
01	0	0	0	7	0	0	0	31	41	0	7	19	0	0	11	43	13	7	0	0	0	37	0	0	7	11	0	59	0	13
03	0	13	19	17	0	7	11	0	0	0	29	41	7	0	13	0	0	11	0	7	17	19	0	47	67	0	7	13	11	0
07	0	11	7	19	0	0	0	0	23	7	13	0	11	0	59	17	7	41	0	0	31	43	19	7	0	0	0	29	47	0
09	0	0	0	0	7	0	23	47	13	53	0	7	0	19	31	11	0	0	7	37	41	13	0	43	0	7	11	0	67	23
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49	17	0	0	7	0	0	0	11	13	19	7	0	97	61	23	0	43	7	11	107	0	13	0	79	7	0	0	101	19	11
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	2 72	75	78	81	84	87	90	93	96	2 99	3 02	05	08	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59
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49	37	7	0	0	13	0	137	23	7	53	11	103	0	29	157	7	0	13	0	0	113	11	7	0	67	0	71	0	0	7
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	4 50	53	56	59	62	65	68	71	74	77	80	83	86	89	92	95	4 93	5 01	04	07	10	13	16	19	22	25	28	31	34	37
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	7 22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88	91	94	7 97	3 00	03	06	09
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67	23	41	7	0	31	13	163	0	11	7	17	0	29	257	19	0	7	281	13	11	67	47	0	7	97	0	43	17	0	0
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	9 01	04	07	10	13	16	19	22	25	23	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
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71	0	13	11	17	23	0	7	71	0	239	19	137	0	7	13	0	0	283	29	0	7	269	73	0	11	0	101	7	79	19
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87	17	7	0	67	11	263	71	0	7	0	0	13	0	97	19	7	0	17	103	0	73	0	7	0	13	0	11	0	29	7
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93	7	17	11	0	0	13	19	7	0	0	29	173	0	11	7	0	0	0	13	59	0	7	0	83	11	19	233	61	7	0
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Log Γ (*n*).

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1.02	51279	48916	46561	44212	41870	39535	37207	34886	32572	30265
1.03	27964	25671	23384	21104	18831	16564	14305	12052	09806	07567
1.04	05334	03108	00889	$\overline{9}8677$	$\overline{9}6471$	$\overline{9}4273$	$\overline{9}2080$	$\overline{8}9895$	$\overline{8}7716$	$\overline{8}5544$
1.05	9.9883379	81220	79068	76922	74783	72651	70525	68406	66294	64188
1.06	62089	59996	57910	55830	53757	51690	49630	47577	45530	43489
1.07	41469	39428	37407	35392	33384	31382	29387	27398	25415	23449
1.08	21469	19506	17549	15599	13655	11717	09785	07860	05941	04029
1.09	02123	00223	$\overline{9}8329$	$\overline{9}6442$	$\overline{9}4561$	$\overline{9}2686$	$\overline{9}0818$	$\overline{8}8956$	$\overline{8}7100$	$\overline{8}5250$
1.10	9.9783407	81570	79738	77914	76095	74283	72476	70676	68882	67095
1.11	65313	63538	61768	60005	58248	56497	54753	53014	51281	49555
1.12	47834	46120	44411	42709	41013	39323	37638	35960	34288	32622
1.13	30962	29308	27659	26017	24381	22751	21126	19508	17896	16289
1.14	14689	13094	11505	09922	08345	06774	05209	03650	02096	00549
1.15	9.9699007	97471	95941	94417	92898	91386	89879	88378	86883	85393
1.16	83910	82432	80960	79493	78033	76578	75129	73686	72248	70816
1.17	69390	67969	66554	65145	63742	62344	60952	59566	58185	56810
1.18	55440	54076	52718	51366	50019	48677	47341	46011	44687	43368
1.19	42054	40746	39444	38147	36856	35570	34290	33016	31747	30483
1.20	29225	27973	26725	25484	24248	23017	21792	20573	19358	18150
1.21	16946	15748	14556	13369	12188	11011	09841	08675	07515	06361
1.22	05212	04068	02930	01796	00669	$\overline{9}9546$	$\overline{9}8430$	$\overline{9}7318$	$\overline{9}6212$	$\overline{9}5111$
1.23	9.9594015	92925	91840	90760	89685	88616	87553	86494	85441	84393
1.24	83350	82313	81280	80253	79232	78215	77204	76198	75197	74201
1.25	73211	72226	71246	70271	69301	68337	67377	66423	65474	64530
1.26	63592	62658	61730	60806	59888	58975	58067	57165	56267	55374
1.27	54487	53604	52727	51855	50988	50126	49268	48416	47570	46728
1.28	45891	45059	44232	43410	42593	41782	40975	40173	39376	38585
1.29	37798	37016	36239	35467	34700	33938	33181	32439	31682	30940
1.30	30203	29470	28743	28021	27303	26590	25883	25180	24482	23789
1.31	23100	22417	21739	21065	20396	19732	19073	18419	17770	17125
1.32	16485	15850	15220	14595	13975	13359	12748	12142	11540	10944
1.33	10353	09766	09184	08606	08034	07466	06903	06344	05791	05242
1.34	04698	04158	03624	03094	02568	02048	01532	01021	00514	00012
1.35	9.9499515	99023	98535	98052	97573	97100	96630	96166	95706	95251
1.36	94800	94355	93913	93477	93044	92617	92194	91776	91362	90953
1.37	90549	90149	89754	89363	88977	88595	88218	87846	87478	87115
1.38	86756	86402	86052	85707	85366	85030	84698	84371	84049	83731
1.39	83417	83108	82803	82503	82208	81916	81630	81348	81070	80797
1.40	80528	80263	80003	79748	79497	79250	79008	78770	78537	78308
1.41	78084	77864	77648	77437	77230	77027	76829	76636	76446	76261
1.42	76081	75905	75733	75565	75402	75243	75089	74939	74793	74652
1.43	74515	74382	74254	74130	74010	73894	73783	73676	73574	73476
1.44	73382	73292	73207	73125	73049	72976	72908	72844	72784	72728
1.45	72677	72630	72587	72549	72514	72484	72459	72437	72419	72406
1.46	72397	72393	72392	72396	72404	72416	72432	72452	72477	72506
1.47	72539	72576	72617	72662	72712	72766	72824	72886	72952	73022
1.48	73097	73175	73258	73345	73436	73531	73630	73734	73841	73953
1.49	74068	74188	74312	74440	74572	74708	74848	74992	75141	75293

NOTE.—This table is taken from Vol. II. of Legendre's work, and not from Vol. I., as stated in the Preface: the numbers given in Vol. I. being inaccurate in the seventh decimal place. In Vol II. the values are given to twelve places of decimals. The figure here printed in the seventh place is

n	0	1	2	3	4	5	6	7	8	9
1.50	9.9475449	75610	75774	75943	76116	76292	76473	76658	76847	77040
1.51	77237	77438	77642	77851	78064	78281	78502	78727	78956	79189
1.52	79426	79667	79912	80161	80414	80671	80932	81196	81465	81738
1.53	82015	82295	82580	82868	83161	83457	83758	84062	84370	84682
1.54	84998	85318	85642	85970	86302	86638	86977	87321	87668	88019
1.55	88374	88733	89096	89463	89834	90208	90587	90969	91355	91745
1.56	92139	92537	92938	93344	93753	94166	94583	95004	95429	95857
1.57	96289	96725	97165	97609	98056	985[illegible]8	98963	99422	99885	$\overline{0}0351$
1.58	9.9500822	01296	01774	02255	02741	03230	03723	04220	04720	05225
1.59	05733	06245	06760	07280	07803	08330	08860	09395	09933	10475
1.60	11020	11569	12122	12679	13240	13804	14372	14943	15519	16098
1.61	16680	17267	17857	18451	19048	19650	20254	20862	21475	22091
1.62	22710	23333	23960	24591	25225	25863	26504	27149	27798	28451
1.63	29107	29767	30430	31097	31767	32442	33120	33801	34486	35175
1.64	35867	36563	37263	37966	38673	39383	40097	40815	41536	42260
1.65	42989	43721	44456	45195	45938	46684	47434	48187	48944	49704
1.66	50468	51236	52007	52782	53560	54342	55127	55916	56708	57504
1.67	58303	59106	59913	60723	61536	62353	63174	63998	64826	65656
1.68	66491	67329	68170	69015	69864	70716	71571	72430	73293	74159
1.69	75028	75901	76777	77657	78540	79427	80317	81211	82108	83008
1.70	83912	84820	85731	86645	87563	88484	89409	90337	91268	92203
1.71	93141	94083	95028	95977	96929	97884	98843	99805	$\overline{0}0771$	$\overline{0}1740$
1.72	9.9602712	03688	04667	05650	06636	07625	08618	09614	10613	11616
1.73	12622	13632	14645	15661	16681	17704	18730	19760	20793	21830
1.74	22869	23912	24959	26009	27062	28118	29178	30241	31308	32377
1.75	33451	34527	35607	36690	37776	38866	39959	41055	42155	43258
1.76	44364	45473	46586	47702	48821	49944	51070	52200	53331	54467
1.77	55606	56749	57894	59043	60195	61350	62509	63671	64836	66004
1.78	67176	68351	69529	70710	71895	73082	74274	75468	76665	77866
1.79	79070	80277	81488	82701	83918	85138	86361	87588	88818	90051
1.80	91287	92526	93768	95014	96263	97515	98770	$\overline{0}0029$	$\overline{0}1291$	$\overline{0}2555$
1.81	9.9703823	05095	06369	07646	08927	10211	11498	12788	14082	15378
1.82	16678	17981	19287	20596	21908	23224	24542	25864	27189	28517
1.83	29848	31182	32520	33860	35204	36551	37900	39254	40610	41969
1.84	43331	44697	46065	47437	48812	50190	51571	52955	54342	55733
1.85	57126	58522	59922	61325	62730	64140	65551	66966	68384	69805
1.86	71230	72657	74087	75521	76957	78397	79839	81285	82734	84186
1.87	85640	87098	88559	90023	91490	92960	94433	95910	97389	98871
1.88	9.9800356	01844	03335	04830	06327	07827	09331	10837	12346	13859
1.89	15374	16893	18414	19939	21466	22996	24530	26066	27606	29148
1.90	30693	32242	33793	35348	36905	38465	40028	41595	43164	44736
1.91	46311	47890	49471	51055	52642	54232	55825	57421	59020	60622
1.92	62226	63834	65445	67058	68675	70294	71917	73542	75170	76802
1.93	78436	80073	81713	83356	85002	86651	88302	89957	91614	93275
1.94	94938	96605	98274	99946	$\overline{0}1621$	$\overline{0}3299$	$\overline{0}4980$	$\overline{0}6663$	$\overline{0}8350$	$\overline{1}0039$
1.95	9.9911732	13427	15125	16826	18530	20237	21947	23659	25375	27093
1.96	28815	30539	32266	33995	35728	37464	39202	40943	42688	44435
1.97	46185	47937	49693	51451	53213	54977	56744	58513	60286	62062
1.98	63840	65621	67405	69192	70982	72774	74570	76368	78169	79972
1.99	81779	83588	85401	87216	89034	90854	92678	94504	96333	98165

the one nearest to the true value whether in excess or defect. This table, and the table of Least Factors, have each been subjected to two complete and independent revisions before finally printing off.

ALGEBRA.

FACTORS.

1 $$a^2-b^2 = (a-b)(a+b).$$
2 $$a^3-b^3 = (a-b)(a^2+ab+b^2).$$
3 $$a^3+b^3 = (a+b)(a^2-ab+b^2).$$

And generally,

4 $a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+\ldots+b^{n-1})$ always.

5 $a^n-b^n = (a+b)(a^{n-1}-a^{n-2}b+\ldots-b^{n-1})$ if n be even.

6 $a^n+b^n = (a+b)(a^{n-1}-a^{n-2}b+\ldots+b^{n-1})$ if n be odd.

---

7 $(x+a)(x+b) = x^2+(a+b)x+ab.$

8 $(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2 +(bc+ca+ab)x+abc.$

9 $(a+b)^2 = a^2+2ab+b^2.$

10 $(a-b)^2 = a^2-2ab+b^2.$

11 $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b).$

12 $(a-b)^3 = a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b).$

Generally,

$$(a\pm b)^7=a^7\pm 7a^6b+21a^5b^2\pm 35a^4b^3+35a^3b^4\pm 21a^2b^5+7ab^6\pm b^7.$$

Newton's Rule for forming the coefficients: *Multiply any coefficient by the index of the leading quantity, and divide by the number of terms to that place to obtain the coefficient of the term next following.* Thus $21\times 5\div 3$ gives 35, the following coefficient in the example given above. See also (125).

To square a polynomial: *Add to the square of each term twice the product of that term and every term that follows it.*

Thus, $$(a+b+c+d)^2$$
$$= a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2.$$

13 $a^4+a^2b^2+b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$.

14 $a^4+b^4 = (a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)$.

15 $\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2,\quad \left(x+\frac{1}{x}\right)^3 = x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$.

16 $(a+b+c)^2 = a^2+b^2+c^2+2bc+2ca+2ab$.

17 $(a+b+c)^3 = a^3+b^3+c^3+3(b^2c+bc^2+c^2a+ca^2$
$+a^2b+ab^2)+6abc$.

Observe that in an algebraical equation *the sign of any letter may be changed throughout*, and thus a new formula obtained, it being borne in mind that an *even* power of a negative quantity is positive. For example, by changing the sign of c in (16), we obtain

$$(a+b-c)^2 = a^2+b^2+c^2-2bc-2ca+2ab.$$

18 $a^2+b^2-c^2+2ab = (a+b)^2-c^2 = (a+b+c)(a+b-c)$
by (1).

19 $a^2-b^2-c^2+2bc = a^2-(b-c)^2 = (a+b-c)(a-b+c)$.

20 $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

21 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+a^3+b^3+c^3$
$= (a+b+c)(a^2+b^2+c^2)$.

22 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+3abc$
$= (a+b+c)(bc+ca+ab)$.

23 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+2abc=(b+c)(c+a)(a+b)$

24 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b-2abc-a^3-b^3-c^3$
$= (b+c-a)(c+a-b)(a+b-c)$.

25 $bc^2-b^2c+ca^2-c^2a+ab^2-a^2b = (b-c)(c-a)(a-b)$.

26 $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

27 $x^3+2x^2y+2xy^2+y^3 = (x+y)(x^2+xy+y^2)$.

Generally for the division of $(x+y)^n-(x^n+y^n)$ by x^2+xy+y^2 see (545).

MULTIPLICATION AND DIVISION,

BY THE METHOD OF DETACHED COEFFICIENTS.

28 Ex. 1: $(a^4-3a^2b^2+2ab^3+b^4)\times(a^3-2ab^2-2b^3)$.

$$\begin{array}{l}
1+0-3+2+1\\
1+0-2-2\\
\hline
1+0-3+2+1\\
\quad\;\; -2-0+6-4-2\\
\qquad\quad -2-0+6-4-2\\
\hline
1+0-5+0+7+2-6-2
\end{array}$$

Result $a^7-5a^5b^2+7a^3b^4+2a^2b^5-6ab^6-2b^7$.

Ex. 2: $(x^7-5x^5+7x^3+2x^2-6x-2)\div(x^4-3x^2+2x+1)$.

$$\begin{array}{l}
1+0-3+2+1\;)\;1+0-5+0+7+2-6-2\;(\;1+0-2-2\\
\qquad\qquad\quad\; -1-0+3-2-1\\
\qquad\qquad\qquad\quad 0-2-2+6+2-6\\
\qquad\qquad\qquad\quad\;\; +2+0-6+4+2\\
\qquad\qquad\qquad\qquad\quad -2+0+6-4-2\\
\qquad\qquad\qquad\qquad\quad +2+0-6+4+2
\end{array}$$

Result x^3-2x-2.

Synthetic Division.

Ex. 3: Employing the last example, the work stands thus,

$$\begin{array}{r|l}
 & 1+0-5+0+7+2-6-2\\
-0 & \quad 0+0+0+0\\
+3 & \qquad +3+0-6-6\\
-2 & \qquad\quad -2+0+4+4\\
-1 & \qquad\qquad -1+0+2+2\\
\hline
 & 1+0-2-2
\end{array}$$

Result x^3-2x-2. [See also (248).

Note that, in all operations with detached coefficients, the result must be written out in successive powers of the quantity which stood in its successive powers in the original expression.

INDICES.

29 Multiplication: $a^{\frac{1}{3}} \times a^{\frac{1}{2}} = a^{\frac{1}{3}+\frac{1}{2}} = a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$;

$$a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{1}{m}+\frac{1}{n}} = a^{\frac{m+n}{mn}}, \text{ or } \sqrt[mn]{a^{m+n}}.$$

Division: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}$, or $\sqrt[6]{a}$;

$$a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{1}{n}-\frac{1}{m}} = a^{\frac{m-n}{mn}}, \text{ or } \sqrt[mn]{a^{m-n}}.$$

Involution: $(a^{\frac{2}{3}})^{\frac{1}{2}} = a^{\frac{2}{3}\times\frac{1}{2}} = a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

Evolution: $\sqrt[7]{a^{\frac{2}{3}}} = a^{\frac{2}{3}\times\frac{1}{7}} = a^{\frac{2}{21}}$, or $\sqrt[21]{a^2}$.

$$a^{-n} = \frac{1}{a^n}, \quad a^0 = 1.$$

HIGHEST COMMON FACTOR.

30 Rule.—To find the highest common factor of two expressions: *Divide the one which is of the highest dimension by the other, rejecting first any factor of either expression which is not also a factor of the other. Operate in the same manner upon the remainder and the divisor, and continue the process until there is no remainder. The last divisor will be the highest common factor required.*

31 Example.—To find the H. C. F. of

$3x^5 - 10x^3 + 15x + 8$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.

	1− 2− 6+ 4+13+ 6	3+0−10+ 0+15+ 8	3
	3	−3+6+18−12−39−18	
1	3− 6−18+12+39+18	2) 6+ 8−12−24−10	
	−3− 4+ 6+12+ 5	3+ 4− 6−12− 5	
	2) −10−12+24+44+18	−3− 9− 9− 3	
	− 5− 6+12+22+ 9	− 5−15−15− 5	
	3	+ 5+15+15+ 5	
5	−15−18+36+66+27		
	+15+20−30−60−25		
	2) 2+ 6+ 6+ 2	Result H. C. F. $= x^3+3x^2+3x+1$.	
	1+ 3+ 3+ 1		

32 Otherwise.—To form the H. C. F. of two or more algebraical expressions: *Separate the expressions into their simplest factors. The H. C. F. will be the product of the factors common to all the expressions, taken in the lowest powers that occur.*

LOWEST COMMON MULTIPLE.

33 *The L. C. M. of two quantities is equal to their product divided by the H. C. F.*

34 Otherwise.—To form the L. C. M. of two or more algebraical expressions: *Separate them into their simplest factors. The L. C. M. will be the product of all the factors that occur, taken in the highest powers that occur.*

EXAMPLE.—The H. C. F. of $a^2(b-x)^5c^7d$ and $a^3(b-x)^2c^4e$ is $a^2(b-x)^2c^4$; and the L. C. M. is $a^3(b-x)^5c^7de$.

EVOLUTION.

To extract the Square Root of

$$a^2 - \frac{3a\sqrt{a}}{2} - \frac{3\sqrt{a}}{2} + \frac{41a}{16} + 1.$$

Arranging according to powers of a, and reducing to one denominator, the expression becomes $\dfrac{16a^2 - 24a^{\frac{3}{2}} + 41a - 24a^{\frac{1}{2}} + 16}{16}$.

35 Detaching the coefficients, the work is as follows:—

$$\begin{array}{rl|l}
 & & 16-24+41-24+16\ (\,4-3+4 \\
 & & 16 \\
8-3 & & -24+41 \\
-3 & & 24-\ 9 \\
\hline
8-6+4 & & \qquad 32-24+16 \\
 & & \qquad -32+24-16
\end{array}$$

Result $\dfrac{4a - 3a^{\frac{1}{2}} + 4}{4} = a - \frac{3}{4}\sqrt{a} + 1$

To extract the Cube Root of

37 $$8x^6-36x^5\sqrt{y}+66x^4y-63x^3y\sqrt{y}+33x^2y^2-9x^2\sqrt{y}+y^3.$$

The terms here contain the successive powers of x and $\sqrt{y}$; therefore, detaching the coefficients, the work will be as follows:—

I.	II.	III.
$6-3$	12	$8-36+66-63+33-9+1\ (\ 2-3+1$
-6	$-18+\ 9$	-8
$6-9+1$	$12-18+\ 9$	$-36+66-63+33-9+1$
	$+\ 9$	$+36-54+27$
	$12-36+27$	$12-36+33-9+1$
	$6-9+1$	$-12+36-33+9-1$
	$12-36+33-9+1$	

Result $\quad 2x^2-3x\sqrt{y}+y.$

Explanation.—The cube root of 8 is 2, the first term of the result.

Place $3\times 2=6$ in the first column I., $3\times 2^2=12$ in column II., and $2^3=8$ in III., changing its sign for subtraction.

$-36\div 12=-3$, the second term of the result.

Put -3 in I.; $(6-3)\times(-3)$ gives $-18+9$ for II.

$(12-18+9)\times 3$ (changing sign) gives $36-54+27$ for III. Then add.

Put twice (-3), the term last found, in I., and the square of it in II. Add the two last rows in I., and the three last in II.

$12\div 12$ gives 1, the third term of the result.

Put 1 in col. I., $(6-9+1)\times 1$ gives $6-9+1$ for col. II.

$(12-36+33-9+1)\times 1$ gives the same for III. Change the signs, and add, and the work is finished.

The foregoing process is but a slight variation of Horner's rule for solving an equation of any degree. See (533).

Transformations frequently required.

38 If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$ .... [68.

39 If $\left.\begin{array}{l} x+y=a \\ \text{and } x-y=b \end{array}\right\}$, then $\left\{\begin{array}{l} x=\tfrac{1}{2}(a+b). \\ x=\tfrac{1}{2}(a-b). \end{array}\right.$

---

40 $$(x+y)^2+(x-y)^2=2(x^2+y^2).$$

41 $$(x+y)^2-(x-y)^2=4xy.$$

42 $$(x+y)^2 = (x-y)^2 + 4xy.$$

43 $$(x-y)^2 = (x+y)^2 - 4xy.$$

44 EXAMPLES.

$$\frac{2\sqrt{a^2-b^2}+\sqrt{c^2-x^2}}{2\sqrt{a^2-b^2}-\sqrt{c^2-x^2}} = \frac{3\sqrt{a^2-b^2}+\sqrt{c^2-d^2}}{3\sqrt{a^2-b^2}-\sqrt{c^2-d^2}},$$

$$\frac{\sqrt{c^2-x^2}}{2\sqrt{a^2-b^2}} = \frac{\sqrt{c^2-d^2}}{3\sqrt{a^2-b^2}}$$ ....[38.

$$9(c^2-x^2) = 4(c^2-d^2),$$

$$x = \sqrt{5c^2+4d^2}.$$

To simplify a compound fraction, as

$$\frac{\dfrac{1}{a^2-ab+b^2}+\dfrac{1}{a^2+ab+b^2}}{\dfrac{1}{a^2-ab+b^2}-\dfrac{1}{a^2+ab+b^2}},$$

multiply the numerator and denominator by the L. C. M. of all the smaller denominators.

Result $$\frac{(a^2+ab+b^2)+(a^2-ab+b^2)}{(a^2+ab+b^2)-(a^2-ab+b^2)} = \frac{a^2+b^2}{ab}.$$

QUADRATIC EQUATIONS.

45 If $ax^2+bx+c=0$, $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

46 If $ax^2+2bx+c=0$; that is, if the coefficient of x be an even number, $x = \dfrac{-b\pm\sqrt{b^2-ac}}{a}$.

47 *Method of solution without the formula.*

Ex.: $$2x^2-7x+3=0.$$

Divide by 2, $$x^2-\frac{7}{2}x+\frac{3}{2}=0.$$

Complete the square, $x^2-\frac{7}{2}x+\left(\frac{7}{4}\right)^2=\frac{49}{16}-\frac{3}{2}=\frac{25}{16}.$

Take square root, $x-\frac{7}{4}=\pm\frac{5}{4},$

$$x=\frac{7\pm5}{4}=3 \quad\text{or}\quad \frac{1}{2}.$$

48 Rule for "completing the square" of an expression like $x^2-\frac{7}{2}x$: *Add the square of half the coefficient of x.*

49 The solution of the foregoing equation, employing formula (45), is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{7\pm\sqrt{49-24}}{4}=\frac{7\pm5}{4}=3 \quad\text{or}\quad \frac{1}{2}.$$

THEORY OF QUADRATIC EXPRESSIONS.

If α, β be the roots of the equation $ax^2+bx+c=0$, then

50 $$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

51 Sum of roots $\alpha+\beta=-\frac{b}{a}.$

52 Product of roots $\alpha\beta=\frac{c}{a}.$

Condition for the existence of equal roots—

53 b^2-4ac must vanish.

54 The solution of equations in one unknown quantity may sometimes be simplified by changing the quantity sought.

Ex. (1): $$2x+\frac{3x-1}{3x+1}+\frac{18x+6}{6x^2+5x-1}=14 \quad\ldots\ldots\ldots\ldots \quad (1).$$

$$\frac{6x^2+5x-1}{3x+1}+\frac{6(3x+1)}{6x^2+5x-1}=14.$$

Put $$y=\frac{6x^2+5x-1}{3x+1} \quad\ldots\ldots\ldots\ldots \quad (2).$$

thus $$y + \frac{6}{y} = 14.$$
$$y^2 - 14y + 6 = 0.$$
y having been determined from this quadratic, x is afterwards found from (2).

---

55 Ex. 2: $$x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4.$$
$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6.$$
Put $x + \dfrac{1}{x} = y$, and solve the quadratic in y.

---

56 Ex. 3: $$x^2 + x + \frac{3}{2}\sqrt{2x^2 + x + 2} = \frac{x}{2} + 1.$$
$$2x^2 + x + 3\sqrt{2x^2 + x + 2} = 2,$$
$$2x^2 + x + 2 + 3\sqrt{2x^2 + x + 2} = 4.$$
Put $\sqrt{2x^2 + x + 2} = y$, and solve the quadratic
$$y^2 + 3y = 4.$$

---

57 Ex. 4: $$\sqrt[3]{x^n} + \frac{2}{3\sqrt[3]{x^n}} = \frac{16}{3}x^{-n}.$$
$$x^{\frac{4n}{3}} + \frac{2}{3}x^{\frac{2n}{3}} = \frac{16}{3}.$$
A quadratic in $$y = x^{\frac{2n}{3}}.$$

---

58 *To find Maxima and Minima values by means of a Quadratic Equation.*

Ex.—Given $$y = 3x^2 + 6x + 7,$$
to find what value of x will make y a maximum or minimum.

Solve the quadratic equation
$$3x^2 + 6x + 7 - y = 0.$$
Thus $$x = \frac{-3 \pm \sqrt{3y - 12}}{3}. \qquad [46.$$

In order that x may be a real quantity, we must have $3y$ not less than 12; therefore 4 is a minimum value of y, and the value of x which makes y a minimum is -1.

SIMULTANEOUS EQUATIONS.

General solution with two unknown quantities.

Given

59 $$\left.\begin{aligned} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{aligned}\right\}, \quad x = \frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}, \quad y = \frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}.$$

General solution with three unknown quantities.

60 Given $$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\},$$

$$x = \frac{d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)}{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)},$$

and symmetrical forms for y and z.

Methods of solving simultaneous equations between two unknown quantities x and y.

61 I. *By substitution.—Find one unknown in terms of the other from one of the two equations, and substitute this value in the remaining equation. Then solve the resulting equation.*

Ex.: $$\left.\begin{aligned} x+5y &= 23 \;\ldots\ldots\; (1) \\ 7y &= 28 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

From (2), $y = 4$. Substitute in (1); thus

$$x+20 = 23, \quad x = 3.$$

62 II. *By the method of Multipliers.*

Ex.: $$\left.\begin{aligned} 3x+5y &= 36 \;\ldots\ldots\; (1) \\ 2x-3y &= \;\;5 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

Eliminate x by multiplying eq. (1) by 2, and (2) by 3; thus

$$6x+10y = 72,$$
$$6x-\;9y = 15,$$
$$19y = 57, \text{ by subtraction,}$$
$$y = 3;$$
$$\therefore\quad x = 7, \text{ by substitution in eq. (2).}$$

63 III. *By changing the quantities sought.*

Ex. 1:
$$\left.\begin{aligned} x-y &= 2 \ldots\ldots (1) \\ x^2-y^2+x+y &= 30 \ldots\ldots (2) \end{aligned}\right\}.$$

Let $x+y = u, \quad x-y = v.$

Substitute these values in (1) and (2),
$$\left.\begin{aligned} v &= 2 \\ uv+u &= 30 \end{aligned}\right\};$$
$$\therefore \quad 2u+u = 30,$$
$$u = 10;$$
$$\therefore \quad x+y = 10,$$
$$x-y = 2.$$

From which $x = 6$ and $y = 4$.

---

64 Ex. 2:
$$\left.\begin{aligned} 2\frac{x+y}{x-y}+10\frac{x-y}{x+y} &= 9 \ldots\ldots (1) \\ x^2+7y^2 &= 64 \ldots\ldots (2) \end{aligned}\right\}.$$

Substitute z for $\frac{x+y}{x-y}$ in (1);
$$\therefore \quad 2z+\frac{10}{z} = 9;$$
$$2z^2-9z+10 = 0.$$

From which $z = \frac{5}{2}$ or 2,
$$\frac{x+y}{x-y} = 2 \text{ or } \frac{5}{2}.$$

From which $x = 3y$ or $\frac{7}{3}y$.

Substitute in (2); thus $y = 2$ and $x = 6$,

or $y = \frac{6}{\sqrt{7}}$ and $x = 2\sqrt{7}$.

---

65 Ex. 3:
$$\left.\begin{aligned} 3x+5y &= xy \ldots\ldots (1) \\ 2x+7y &= 3xy \ldots\ldots (2) \end{aligned}\right\}.$$

Divide each quantity by xy;
$$\left.\begin{aligned} \frac{3}{y}+\frac{5}{x} &= 1 \ldots\ldots (3) \\ \frac{2}{y}+\frac{7}{x} &= 3 \ldots\ldots (4) \end{aligned}\right\}.$$

Multiply (3) by 2, and (4) by 3, and by subtraction y is eliminated.

66 IV. *By substituting $y = tx$, when the equations are homogeneous in the terms which contain x and y.*

Ex. 1: $$\left.\begin{aligned} 52x^2+7xy &= 5y^2 \;\ldots\ldots (1) \\ 5x-3y &= 17 \;\ldots\ldots (2) \end{aligned}\right\}.$$

From (1), and, from (2), $$\left.\begin{aligned} 52x^2+7tx^2 &= 5t^2x^2 \ldots\ldots (3) \\ 5x-3tx &= 17 \;\ldots\ldots (4) \end{aligned}\right\}.$$

(3) gives $$52+7t = 5t^2,$$

a quadratic equation from which t must be found, and its value substituted in (4).

x is thus determined; and then y from $y = tx$.

67 Ex. 2: $$\left.\begin{aligned} 2x^2+xy+3y^2 &= 16 \ldots\ldots (1) \\ 3y-2x &= 4 \ldots\ldots (2) \end{aligned}\right\}.$$

From (1), by putting $y = tx$,

$$\left.\begin{aligned} x^2(2+t+3t^2) &= 16 \ldots\ldots (3) \\ x(3t-2) &= 4 \ldots\ldots (4) \end{aligned}\right\};$$

from (2),

squaring, $$x^2(9t^2-12t+4) = 16;$$

$$\therefore \quad 9t^2-12t+4 = 2+t+3t^2,$$

a quadratic equation for t.

t being found from this, equation (4) will determine x; and finally $y = tx$.

RATIO AND PROPORTION.

68 If $a : b :: c : d$; then $ad = bc$, and $\frac{a}{b} = \frac{c}{d}$;

$$\frac{a+b}{b} = \frac{c+d}{d}; \quad \frac{a-b}{b} = \frac{c-d}{d}; \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

69 If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \&c.$; then $\frac{a}{b} = \frac{a+c+e+\&c.}{b+d+f+\&c.}$.

General theorem.

70 If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \&c. = k$ say, then

$$k = \left\{\frac{pa^n+qc^n+re^n+\&c.}{pb^n+qd^n+rf^n+\&c.}\right\}^{\frac{1}{n}}$$

where p, q, r, &c. are any quantities whatever. Proved as in (71).

71 Rule.—To verify any equation between such proportional quantities: *Substitute for a, c, e, &c., their equivalents kb, kd, kf, &c. respectively, in the given equation.*

Ex.—If $a : b :: c : d$, to show that

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}}.$$

Put kb for a, and kd for c; thus

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{kb-b}}{\sqrt{kd-d}} = \frac{\sqrt{b}\sqrt{k-1}}{\sqrt{d}\sqrt{k-1}} = \frac{\sqrt{b}}{\sqrt{d}};$$

also
$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}} = \frac{\sqrt{kb}-\sqrt{b}}{\sqrt{kd}-\sqrt{d}} = \frac{\sqrt{b}\,(\sqrt{k}-1)}{\sqrt{d}\,(\sqrt{k}-1)} = \frac{\sqrt{b}}{\sqrt{d}}.$$

Identical results being obtained, the proposed equation must be true.

72 If $a : b : c : d : e$ &c., forming a continued proportion,
then $a : c :: a^2 : b^2$, the duplicate ratio of $a : b$,
$a : d :: a^3 : b^3$, the triplicate ratio of $a : b$, and so on.
Also $\sqrt{a} : \sqrt{b}$ is the subduplicate ratio of $a : b$,
$a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is the sesquiplicate ratio of $a : b$.

73 The fraction $\frac{a}{b}$ is made to approach nearer to unity in value, by adding the same quantity to the numerator and denominator. Thus

$$\frac{a+x}{b+x} \text{ is nearer to 1 than } \frac{a}{b} \text{ is.}$$

74 Def.—The ratio compounded of the ratios $a : b$ and $c : d$ is the ratio $ac : bd$.

75 If $a : b :: c : d$, and $a' : b' :: c' : d'$; then, by compounding ratios, $aa' : bb' :: cc' : dd'$.

VARIATION.

76 If $a \propto c$ and $b \propto c$, then $(a \pm b) \propto c$ and $\sqrt{ab} \propto c$.

77 If $\left.\begin{array}{l} a \propto b \\ c \propto d \end{array}\right\}$ (the second line beginning "and"), then $ac \propto bd$ and $\frac{a}{c} \propto \frac{b}{d}$.

78 If $a \propto b$, we may assume $a = mb$, where m is some constant.

ARITHMETICAL PROGRESSION.

General form of a series in A. P.

79 $$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots\ a+(n-1)\,d.$$

$a =$ first term,

$d =$ common difference,

$l =$ last of n terms,

$s =$ sum of n terms; then

80 $$l = a+(n-1)\,d.$$

81 $$s = (a+l)\frac{n}{2}.$$

82 $$s = \{2a+(n-1)\,d\}\frac{n}{2}.$$

Proof.—By writing (79) in reversed order, and adding both series together.

GEOMETRICAL PROGRESSION.

General form of a series in G. P.

83 $$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ ar^{n-1}.$$

$a =$ first term,

$r =$ common ratio,

$l =$ last of n terms,

$s =$ sum of n terms; then

84 $$l = ar^{n-1}.$$

85 $$s = a\frac{r^n-1}{r-1} \quad \text{or} \quad a\frac{1-r^n}{1-r}.$$

If r be less than 1, and n be infinite,

86 $$s = \frac{a}{1-r}, \quad \text{since} \quad r^n = 0.$$

Proof.—(85) is obtained by multiplying (83) by r, and subtracting one series from the other.

HARMONICAL PROGRESSION.

87 a, b, c, d, &c. are in Harm. Prog. when the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. are in Arith. Prog.,

88 Or when $a : b :: a-b : b-c$ is the relation subsisting between any three consecutive terms.

89 n^{th} term of the series $= \dfrac{ab}{(n-1)\,a-(n-2)}$. [87, 80.

90 Approximate sum of n terms of the Harm. Prog. Errata
$\frac{1}{a+d}, \frac{\cancel{1}\,a}{a+2d}, \frac{\cancel{1}\,a^2}{a+3d}$, &c., when d is small compared with a,

$$= \frac{(a+d)^n - a^n}{d\,(a+d)^n}.$$

Proof.—By taking instead the G.P. $\frac{1}{a+d} + \frac{a}{(a+d)^2} + \frac{a^2}{(a+d)^3} + \dots$

---

91 Arithmetic mean between a and $b = \dfrac{a+b}{2}$.

92 Geometric do. $= \sqrt{ab}$.

93 Harmonic do. $= \dfrac{2ab}{a+b}$.

The three means are in continued proportion.

PERMUTATIONS AND COMBINATIONS.

94 The number of permutations of n things taken *all* at a time $= n\,(n-1)\,(n-2) \dots 3 \,.\, 2 \,.\, 1 \equiv n\,!$ or $n^{(n)}$.

Proof by Induction.—Assume the formula to be true for n things. Now take $n+1$ things. After each of these the remaining n things may be arranged in $n\,!$ ways, making in all $n \times n\,!$ [that is $(n+1)\,!$] permutations of $n+1$ things; therefore, &c. See also (233) for the mode of proof by Induction.

95 The number of permutations of n things taken r at a time is denoted by $P(n, r)$.

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1) \equiv n^{(r)}.$$

Proof.—By (94); for $(n-r)$ things are left out of each permutation; therefore $P(n, r) = n! \div (n-r)!$.

Observe that $r =$ the number of factors.

---

96 The number of combinations of n things taken r at a time is denoted by $C(n, r)$.

$$C(n, r) = \frac{n(n-1)(n-2) \ldots (n-r+1)}{1 \,.\, 2 \,.\, 3 \ldots r} \equiv \frac{n^{(r)}}{r!}$$

$$= \frac{n!}{r!\,(n-r)!} = C(n, n-r).$$

For every combination of r things admits of $r!$ permutations; therefore $C(n, r) = P(n, r) \div r!$

97 $C(n, r)$ is greatest when $r = \frac{1}{2}n$ or $\frac{1}{2}(n \pm 1)$, according as n is even or odd.

---

98 The number of homogeneous products of r dimensions of n things is denoted by $H(n, r)$.

$$H(n, r) = \frac{n(n+1)(n+2) \ldots (n+r-1)}{1 \,.\, 2 \ldots r} \equiv \frac{(n+r-1)^{(r)}}{r!}.$$

When r is $> n$, this reduces to

rrata **99**

$$\frac{(r+1)(r+2) \ldots (n+r-1)}{(\underset{n}{\cancel{r}}-1)!}.$$

Proof.—$H(n, r)$ is equal to the number of terms in the product of the expansions by the Bin. Th. of the n expressions $(1-ax)^{-1}$, $(1-bx)^{-1}$, $(1-cx)^{-1}$, &c.

Put $a = b = c = \&c. = 1$. The number will be the coefficient of x^r in $(1-x)^{-n}$. (128, 129.)

100 The number of permutations of n things taken all together, when a of them are alike, b of them alike, c alike, &c.

$$= \frac{n!}{a!\,b!\,c!\,\ldots\,\&c.}.$$

For, if the a things were all different, they would form $a!$ permutations where there is now but one. So of b, c, &c.

---

101 The number of combinations of n things r at a time, in which any p of them will always be found, is

$$= C(n-p,\ r-p).$$

For, if the p things be set on one side, we have to add to them $r-p$ things taken from the remaining $n-p$ things in every possible way.

---

102 Theorem: $C(n-1,\ r-1)+C(n-1,\ r) = C(n,\ r)$.

Proof by Induction; or as follows: Put one out of n letters aside; there are $C(n-1,\ r)$ combinations of the remaining $n-1$ letters r at a time. To complete the total $C(n,\ r)$, we must place with the excluded letter all the combinations of the remaining $n-1$ letters $r-1$ at a time.

---

103 If there be one set of P things, another of Q things, another of R things, and so on; the number of combinations formed by taking one out of each set is $= PQR\ \ldots$ &c., the product of the numbers in the several sets.

For one of the P things will form Q combinations with the Q things. A second of the P things will form Q more combinations; and so on. In all, PQ combinations of two things. Similarly there will be PQR combinations of three things; and so on. This principle is very important.

---

104 On the same principle, if p, q, r, &c. things be taken out of each set respectively, the number of combinations will be the product of the numbers of the separate combinations;

that is, $= C(Pp)\,.\,C(Qq)\,.\,C(Rr)\ \ldots$ &c.

105 The number of combinations of n things taken m at a time, when p of the n things are alike, q of them alike, r of them alike, &c., will be the sum of all the combinations of each possible form of m dimensions, and this is equal to the coefficient of x^m in the expansion of

$$(1+x+x^2+\ldots+x^p)(1+x+x^2+\ldots+x^q)(1+x+x^2+\ldots+x^r)\ldots.$$

106 The total number of possible combinations under the same circumstances, when the n things are taken in all ways, 1, 2, 3 ... n at a time,

$$=(p+1)(q+1)(r+1)\ldots-1.$$

107 The number of permutations when they are taken m at a time in all possible ways will be equal to the product of $m\,!$ and the coefficient of x^m in the expansion of

$$\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^p}{p!}\right\}\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^q}{q!}\right\}\ldots$$

......&c.

SURDS.

108 To reduce $\sqrt[3]{2808}$. Decompose the number into its prime factors by (360); thus,

$$\sqrt[3]{2808}=\sqrt[3]{2^3.3^3.13}=6\sqrt[3]{13},$$

$$\sqrt[3]{a^{15}b^{10}c^8}=a^{\frac{15}{3}}b^{\frac{10}{3}}c^{\frac{8}{3}}=a^5b^3c^2b^{\frac{1}{3}}c^{\frac{2}{3}}=a^5b^3c^2\sqrt[3]{bc^2}.$$

109 To bring $5\sqrt[4]{3}$ to an entire surd.

$$5\sqrt[4]{3}=\sqrt[4]{5^4.3}=\sqrt[4]{1875},$$

$$x^{\frac{2}{3}}y^{\frac{1}{5}}z^{\frac{1}{2}}=x^{\frac{20}{30}}y^{\frac{6}{30}}z^{\frac{15}{30}}=\sqrt[30]{x^{20}y^6z^{15}}.$$

110 *To rationalise fractions having surds in their denominators.*

$$\frac{1}{\sqrt{7}}=\frac{\sqrt{7}}{7};\qquad \frac{1}{\sqrt[3]{7}}=\frac{\sqrt[3]{49}}{\sqrt[3]{7\times 49}}=\frac{\sqrt[3]{49}}{7}.$$

111 $$\frac{3}{9-\sqrt{80}} = \frac{3\,(9+\sqrt{80})}{81-80} = 3\,(9+\sqrt{80}),$$

since $$(9-\sqrt{80})\,(9+\sqrt{80}) = 81-80, \text{ by (1)}.$$

---

112 $$\frac{1}{1+2\sqrt{3}-\sqrt{2}} = \frac{1+2\sqrt{3}+\sqrt{2}}{(1+2\sqrt{3})^2-2} = \frac{1+2\sqrt{3}+\sqrt{2}}{11+4\sqrt{3}}$$
$$= \frac{(1+2\sqrt{3}+\sqrt{2})\,(11-4\sqrt{3})}{73}.$$

---

113 $$\frac{1}{\sqrt[3]{3}-\sqrt{2}} = \frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{2}}}.$$

Put $3^{\frac{1}{3}} = x$, $2^{\frac{1}{2}} = y$, and take 6 the L.C.M. of the denominators 2 and 3, then

$$\frac{1}{x-y} = \frac{x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5}{x^6-y^6}, \text{ by (4)};$$

therefore $$\frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{2}}} = \frac{3^{\frac{5}{3}}+3^{\frac{4}{3}}2^{\frac{1}{2}}+3^{\frac{3}{3}}2^{\frac{2}{2}}+3^{\frac{2}{3}}2^{\frac{3}{2}}+3^{\frac{1}{3}}2^{\frac{4}{2}}+2^{\frac{5}{2}}}{3^{\frac{6}{3}}-2^{\frac{6}{2}}}$$
$$= 3\sqrt[3]{9}+3\sqrt[6]{72}+6+2\sqrt[6]{648}+4\sqrt[3]{3}+4\sqrt{2}.$$

---

114 $\dfrac{1}{\sqrt[3]{3}+\sqrt{2}}$. Here the result will be the same as in the last example if the signs of the even terms be changed. [See 5.

---

115 A surd cannot be partly rational; that is, $\sqrt{a}$ cannot be equal to $\sqrt{b} \pm c$. Proved by squaring.

116 The product of two unlike squares is irrational;

$$\sqrt{7}\times\sqrt{3} = \sqrt{21}, \text{ an irrational quantity.}$$

117 The sum or difference of two unlike surds cannot produce a single surd; that is, $\sqrt{a}+\sqrt{b}$ cannot be equal to $\sqrt{c}$. By squaring.

118 If $a+\sqrt{m} = b+\sqrt{n}$; then $a = b$ and $m = n$.

Theorems (115) to (118) are proved indirectly.

---

119 If $$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y},$$

then $$\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}.$$

By squaring and by (118).

120 To express in two terms $\sqrt{7+2\sqrt{6}}$.

Let $$\sqrt{7+2\sqrt{6}} = \sqrt{x}+\sqrt{y};$$

then $x+y = 7$ .... by squaring and by (118),

and $x-y = \sqrt{7^2-(2\sqrt{6})^2} = \sqrt{49-24} = 5$, by (119);

$$\therefore\ x = 6 \text{ and } y = 1.$$

Result $\sqrt{6}+1$.

General formula for the same—

121 $$\sqrt{a\pm\sqrt{b}} = \sqrt{\tfrac{1}{2}(a+\sqrt{a^2-b})}\pm\sqrt{\tfrac{1}{2}(a-\sqrt{a^2-b})}.$$

Observe that no simplification is effected unless a^2-b is a perfect square.

---

122 To simplify $\sqrt[3]{a+\sqrt{b}}$.

Assume $$\sqrt[3]{a+\sqrt{b}} = x+\sqrt{y}.$$

Let $$c = \sqrt[3]{a^2-b}.$$

Then x must be found by trial from the cubic equation

$$4x^3-3cx = a,$$

and $$y = x^2-c.$$

No simplification is effected unless a^2-b is a perfect cube.

Ex. 1: $$\sqrt[3]{7+5\sqrt{2}} = x+\sqrt{y}.$$

$$c = \sqrt[3]{49-50} = -1.$$

$$4x^3+3x = 7; \quad \therefore\ x = 1,\ y = 2.$$

Result $1+\sqrt{2}$.

Ex. 2: $\sqrt[3]{9\sqrt{3}-11\sqrt{2}} = \sqrt{x}+\sqrt{y}$, two different surds.

Cubing, $9\sqrt{3}-11\sqrt{2} = x\sqrt{x}+3x\sqrt{y}+3y\sqrt{x}+y\sqrt{y}$;

$$\therefore\ \left.\begin{aligned} 9\sqrt{3} &= (x+3y)\sqrt{x} \\ 11\sqrt{2} &= (3x+y)\sqrt{y} \end{aligned}\right\}; \qquad (118)$$

$$\therefore\ x = 3 \text{ and } y = 2.$$

123 To simplify $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{15})}$.

Assume $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{15})} = \sqrt{x}+\sqrt{y}+\sqrt{z}$.

Square, and equate corresponding surds.

Result $\sqrt{3}+\sqrt{4}+\sqrt{5}$.

---

124 To express $\sqrt[n]{A \pm B}$ in the form of two surds, where A and B are one or both quadratic surds and n is odd. Take q such that $q\,(A^2-B^2)$ may be a perfect n^{th} power, say p^n, by (361). Take s and t the nearest integers to $\sqrt[n]{q\,(A+B)^2}$ and $\sqrt[n]{q\,(A-B)^2}$, then

$$\sqrt[n]{A+B} = \frac{1}{2\sqrt[2n]{q}}\left\{\sqrt{s+t+2p} \pm \sqrt{s+t-2p}\right\}.$$

Example: To reduce $\sqrt[5]{89\sqrt{3}+109\sqrt{2}}$.

Here $A = 89\sqrt{3}$, $B = 109\sqrt{2}$,

$$A^2-B^2 = 1; \quad \therefore\ p=1 \text{ and } q=1.$$

$$\left.\begin{aligned}\sqrt[5]{q\,(A+B)^2} &= 9+f\\ \sqrt[5]{q\,(A-B)^2} &= 1-f\end{aligned}\right\}, \quad \begin{aligned}&f \text{ being a proper fraction;}\\ &\therefore\ s=9,\ t=1.\end{aligned}$$

Result $\frac{1}{2}\left(\sqrt{9+1+2} \pm \sqrt{9+1-2}\right) = \sqrt{3}+\sqrt{2}$.

BINOMIAL THEOREM.

125 $(a+b)^n =$

$$a^n + na^{n-1}b + \frac{n\,(n-1)}{2!}a^{n-2}b^2 + \frac{n\,(n-1)\,(n-2)}{3!}a^{n-3}b^3 + \&c.$$

126 General or $(r+1)^{\text{th}}$ term,

$$\frac{n\,(n-1)\,(n-2)\,\ldots\,(n-r+1)}{r!}a^{n-r}b^r$$

127 or $$\frac{n!}{(n-r)!\,r!}a^{n-r}b^r$$

if n be a positive integer.

If b be negative, the signs of the even terms will be changed.

If n be negative the expansion reduces to

128 $$(a+b)^{-n} =$$
$$a^{-n} - na^{-n-1}b + \frac{n(n+1)}{2!}a^{-n-2}b^2 - \frac{n(n+1)(n+2)}{3!}a^{-n-3}b^3 + \&c.$$

129 General term,
$$(-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}a^{-n-r}b^r. \quad [\text{See } 98.$$

Euler's proof.—Let the expansion of $(1+x)^n$, as in (125), be called $f(n)$. Then it may be proved by Induction that the equation $$f(m)\times f(n) = f(m+n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
is true when m and n are integers, and therefore universally true; because the *form* of an algebraical product is not altered by changing the letters involved into fractional or negative quantities. Hence
$$f(m+n+p+\&c.) = f(m)\times f(n)\times f(p), \ \&c.$$

Put $m = n = p = \&c.$ to k terms, each equal $\frac{h}{k}$, and the theorem is proved for a fractional index.

Again, put $-n$ for m in (1); thus, whatever n may be,
$$f(-n)\times f(n) = f(0) = 1,$$
which proves the theorem for a negative index.

---

130 For the greatest term in the expansion of $(a+b)^n$, take $r =$ the integral part of $\frac{(n+1)b}{a+b}$ or $\frac{(n-1)b}{a-b}$, according as n is positive or negative.

But if b be greater than a, and n negative or fractional, the terms increase without limit.

EXAMPLES.

Required the 40th term of $\left(1 - \frac{2x}{3}\right)^{42}$.

Here $r = 39$; $a = 1$; $b = -\frac{2x}{3}$; $n = 42$.

By (127), the term will be
$$\frac{42!}{3!\,39!}\left(-\frac{2x}{3}\right)^{39} = -\frac{42 \cdot 41 \cdot 40}{1 \cdot 2 \cdot 3}\left(\frac{2x}{3}\right)^{39} \text{ by (96).}$$

Required the 31st term of $(a-x)^{-4}$.

Here $r=30$; $b=-x$; $n=-4$.

By (129), the term is

$$(-1)^{30}\frac{4.5.6\ldots30.31.32.33}{1.2.3\ldots30}a^{-34}(-x)^{30}=\frac{31.32.33}{1.2.3}\cdot\frac{x^{30}}{a^{34}}\text{ by (98).}$$

---

131 Required the greatest term in the expansion of $\dfrac{1}{(1+x)^6}$ when $x=\frac{14}{17}$.

$\dfrac{1}{(1+x)^6}=(1+x)^{-6}$. Here $n=6$, $a=1$, $b=x$ in the formula

$$\frac{(n-1)\,b}{a-b}=\frac{5\times\frac{14}{17}}{1-\frac{14}{17}}=23\tfrac{1}{3};$$

therefore $r=23$, by (130), and the greatest term

$$=(-1)^{23}\frac{5.6.7\ldots27}{1.2.3\ldots23}\left(\frac{14}{17}\right)^{23}=-\frac{24.25.26.27}{1.2.3.4}\left(\frac{14}{17}\right)^{23}.$$

---

132 Find the first negative term in the expansion of $(2a+3b)^{\frac{17}{3}}$.

We must take r the first integer which makes $n-r+1$ negative; therefore $r>n+1=\frac{17}{3}+1=6\frac{2}{3}$; therefore $r=7$. The term will be

$$\frac{\frac{17}{3}\cdot\frac{14}{3}\cdot\frac{11}{3}\cdot\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\left(-\frac{1}{3}\right)}{7!}(2a)^{-\frac{4}{3}}(3b)^7\text{ by (126)}$$

$$=-\frac{17.14.11.8.5.2.1}{7!}\,\frac{b^7}{(2a)^{\frac{4}{3}}}.$$

---

133 Required the coefficient of x^{34} in the expansion of $\left(\dfrac{2+3x}{2-3x}\right)^2$.

$$\frac{(2+3x)^2}{(2-3x)^2}=(2+3x)^2(2-3x)^{-2}=\left(\frac{2+3x}{2}\right)^2\left(1-\frac{3}{2}x\right)^{-2}$$

$$=\left(1+3x+\frac{9}{4}x^2\right)\left\{1+2\left(\frac{3x}{2}\right)+\frac{2.3}{1.2}\left(\frac{3x}{2}\right)^2+\ldots\ldots\right.$$

$$\left.\ldots\ldots+33\left(\frac{3x}{2}\right)^{32}+34\left(\frac{3x}{2}\right)^{33}+35\left(\frac{3x}{2}\right)^{34}+\&c.\ldots\ldots\right\},$$

the three terms last written being those which produce x^{34} after multiplying by the factor $(1+3x+\frac{9}{4}x^2)$; for we have

$$\frac{9}{4}x^2\times33\left(\frac{3x}{2}\right)^{32}+3x\times34\left(\frac{3x}{2}\right)^{33}+1\times35\left(\frac{3x}{2}\right)^{34},$$

giving for the coefficient of x^{34} in the result

$$\frac{297}{4}\left(\frac{3}{2}\right)^{32}+102\left(\frac{3}{2}\right)^{33}+35\left(\frac{3}{2}\right)^{34}=306\left(\frac{3}{2}\right)^{32}.$$

The coefficient of x^n will in like manner be $9n\left(\frac{3}{2}\right)^{n-2}$.

134 To write the coefficient of x^{3m+1} in the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{2n+1}$.

The general term is

$$\frac{(2n+1)!}{(2n+1-r)!\,r!}x^{2(2n-r+1)} \cdot \frac{1}{x^{2r}} = \frac{(2n+1)!}{(2n+1-r)!\,r!}x^{4n-4r+2}.$$

Equate $4n-4r+2$ to $3m+1$, thus

$$r = \frac{4n-3m+1}{4}.$$

Substitute this value of r in the general term; the required coefficient becomes

$$\frac{(2n+1)!}{\left[\frac{1}{4}(4n+3m+3)\right]!\,\left[\frac{1}{4}(4n-3m+1)\right]!}.$$

The value of r shows that there is no term in x^{3m+1} unless $\frac{4n-3m+1}{4}$ is an integer.

135 An approximate value of $(1+x)^n$, when x is small, is $1+nx$, by (125), neglecting x^2 and higher powers of x.

136 Ex.—An approximation to $\sqrt[3]{999}$ by Bin. Th. (125) is obtained from the first two or three terms of the expansion of

$$(1000-1)^{\frac{1}{3}} = 10 - \tfrac{1}{3}\,.\,1000^{-\frac{2}{3}} = 10 - \tfrac{1}{300} = 9\tfrac{299}{300} \text{ nearly.}$$

MULTINOMIAL THEOREM.

The general term in the expansion of $(a+bx+cx^2+\&c.)^n$ is

137 $$\frac{n(n-1)(n-2)\ldots(p+1)}{q!\,r!\,s!\,\ldots}a^p b^q c^r d^s \ldots x^{q+2r+3s+\ldots},$$

where $$p+q+r+s+\&c. = n,$$

and the number of terms p, q, r, &c. corresponds to the number of terms in the given multinomial.

p is integral, fractional, or negative, according as n is one or the other.

If n be an integer, (137) may be written

138 $$\frac{n!}{p!\,q!\,r!\,s!}a^p b^q c^r d_s \ldots x^{q+2r+3s}.$$

[Deduced from the Bin. Theor.

Ex. 1.—To write the coefficient of a^3bc^5 in the expansion of $(a+b+c+d)^{10}$. Here put $n=10$, $x=1$, $p=3$, $q=1$, $r=5$, $s=0$ in (138).

Result $$\frac{10\,!}{3\,!\;5\,!} = 7.8.9.10.$$

Ex. 2.—To obtain the coefficient of x^8 in the expansion of

$$(1-2x+3x^2-4x^3)^4.$$

Here, comparing with (137), we have $a=1$, $b=-2$, $c=3$, $d=-4$,

$$p+q+r+s=4,$$
$$q+2r+3s=8,$$

1	0	1	2
0	2	0	2
0	1	2	1
0	0	4	0

The numbers 1, 0, 1, 2 are particular values of p, q, r, s respectively, which satisfy the two equations given above.

0, 2, 0, 2 are another set of values which also satisfy those equations; and the four rows of numbers constitute all the solutions. In forming these rows always try the highest possible numbers on the right first.

Now substitute each set of values of p, q, r, s in formula (138) successively, as under:

$$\frac{4\,!}{2\,!}\,1^1(-2)^0\,3^1(-4)^2 = 576$$
$$\frac{4\,!}{2\,!\;2\,!}\,1^0(-2)^2\,3^0(-4)^2 = 384$$
$$\frac{4\,!}{2\,!}\,1^0(-2)^1\,3^2(-4)^1 = 864$$
$$\frac{4\,!}{4\,!}\,1^0(-2)^0\,3^1(-4)^0 = 81$$

Result 1905

Ex. 3.—Required the coefficient of x^4 in $(1+2x-4x^2-2x^3)^{-\frac{1}{2}}$.

Here $a=1$, $b=2$, $c=-4$, $d=-2$, $n=-\frac{1}{2}$; and the two equations are

$$p+q+r+s=-\tfrac{1}{2},$$
$$q+2r+3s=4,$$

$-\frac{5}{2}$	1	0	1
$-\frac{5}{2}$	0	2	0
$-\frac{7}{2}$	2	1	0
$-\frac{9}{2}$	4	0	0

Employing formula (137), the remainder of the work stands as follows:

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{5}{2}}2^{1}(-4)^{0}(-2)^{1} = -\ 3$$

$$\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{5}{2}}2^{0}(-4)^{2}(-2)^{0} = \quad 6$$

$$\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)1^{-\frac{7}{2}}2^{2}(-4)^{1}(-2)^{0} = \quad 15$$

$$\frac{1}{4!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)1^{-\frac{9}{2}}2^{4}(-4)^{0}(-2)^{0} = \quad \frac{35}{8}$$

Result $22\frac{3}{8}$

139 The number of terms in the expansion of the multinomial $(a+b+c+$ to n terms$)^r$ is the same as the number of homogeneous products of n things of r dimensions. See (97) and (98).

The greatest coefficient in the expansion of $(a+b+c+$ to m terms$)^n$, n being an integer, is

140 $$\frac{n!}{(q!)^m\,(q+1)^{(k)}}, \text{ where } qm+k=n.$$

Proof.—By making the denominator in (138) as small as possible. The notation is explained in (96).

LOGARITHMS.

142 $\log_a N = x$ signifies that $a^x = N$, or

Def.—*The logarithm of a number is the power to which the base must be raised to produce that number.*

143 $$\log_a a = 1, \quad \log 1 = 0.$$

144 $$\log MN = \log M + \log N.$$

$$\log\frac{M}{N} = \log M - \log N.$$

$$\log (M)^n = n\log M.$$

$$\log\sqrt[n]{M} = \frac{1}{n}\log M.$$ [142

145 $$\log_b a = \frac{\log_c a}{\log_c b}.$$

That is—*The logarithm of a number to any base is equal to the logarithm of the number divided by the logarithm of the base*, the two last named logarithms being taken to any the same base at pleasure.

PROOF.—Let $\log_c a = x$ and $\log_c b = y$; then $a = c^x$, $b = c^y$. Eliminate c.

$$c = a^{\frac{1}{x}} = b^{\frac{1}{y}}; \qquad \therefore\ a = b^{\frac{x}{y}}, \qquad \text{that is, } \log_b a = \frac{x}{y}.$$ Q. e. d.

146 $$\log_b a = \frac{1}{\log_a b}.$$ Put $c = a$ in (145).

147 $$\log_{10} N = \frac{\log_e N}{\log_e 10}$$ by (145).

148 $$\frac{1}{\log_e 10} = \cdot 43429448\ldots$$

is called the modulus of the common system of logarithms; that is, the factor which will convert logarithms of numbers calculated to the base e into the corresponding logarithms to the base 10. See (154).

EXPONENTIAL THEOREM.

149 $$a^x = 1 + cx + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \&c.,$$

where $$c = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

PROOF. $a^x = \{1+(a-1)\}^x$. Expand this by Binomial Theorem, and collect the coefficients of x; thus c is obtained. Assume c_2, c_3, &c., as the coefficients of the succeeding powers of x, and with this assumption write out the expansions of a^x, a^y, and a^{x+y}. Form the product of the first two series, which product must be equivalent to the third. Therefore equate the coefficient of x in this product with that in the expansion of a^{x+y}. In the identity so obtained, equate the coefficients of the successive powers of y to determine c_2, c_3, &c.

Let e be that value of a which makes $c=1$, then

150 $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \&c.$$

151 $$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \&c.$$

$$= 2\cdot718281828\ \ldots$$ [See (295).

PROOF.—By making $x=1$ in (150).

152 By making $x=1$ in (149) and $x=c$ in (150), we obtain

$a = e^c$; that is, $c = \log_e a$. Therefore by (149)

154 $$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

155 $$\log(1+x) = \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

156 $$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c.$$ [154

157 $$\therefore\ \log\frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\}.$$

Put $\dfrac{m-1}{m+1}$ for x in (157); thus,

158 $$\log m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \&c.\right\}.$$

Put $\dfrac{1}{2n+1}$ for x in (157); thus,

159 $\log(n+1) - \log n$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c.\right\}.$$

CONTINUED FRACTIONS AND CONVERGENTS.

160 To find convergents to $3{\cdot}14159 = \frac{314159}{100000}$. Proceed as in the rule for H. C. F.

7	100000	314159	3
	99113	300000	
1	887	14159	15
	854	887	
1	33	5289	
	29	4435	
4	4	854	25
	4	66	
		194	
		165	
		29	7
		28	
		1	

The continued fraction is

$$3+\cfrac{1}{7+\cfrac{1}{15+\&c.}}$$

or, as it is more conveniently written,

$$3+\frac{1}{7+}\;\frac{1}{15+}\;\&c.$$

The convergents are formed as follows :—

$$\begin{array}{cccccccc} 3 & 7 & 15 & 1 & 25 & 1 & 7 & 4 \\ \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}, & \frac{9208}{2931}, & \frac{9563}{3044}, & \frac{76149}{24239}, & \frac{314159}{100000}. \end{array}$$

161 Rule.—Write the quotients in a row, and the first two convergents at sight (in the example 3 and $3+\frac{1}{7}$). Multiply the numerator of any convergent by the next quotient, and add the previous numerator. The result is the numerator of the next convergent. Proceed in the same way to determine the denominator. The last convergent should be the original fraction in its lowest terms.

162 *Formula for forming the convergents.*

If $\frac{p_{n-2}}{q_{n-2}}$, $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are any consecutive convergents, and a_{n-2}, a_{n-1}, a_n the corresponding quotients; then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

The n^{th} convergent is therefore

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \equiv F_n.$$

The true value of the continued fraction will be expressed by

163 $$F = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}},$$

in which a'_n is the complete quotient or value of the continued fraction commencing with a_n.

164 $p_n q_{n-1} - p_{n-1} q_n = \pm 1$ alternately, by (162).

The convergents are alternately greater and less than the original fraction, and are always in their lowest terms.

165 The difference between F_n and the true value of the continued fraction is

$$< \frac{1}{q_n q_{n+1}} \quad \text{and} \quad > \frac{1}{q_n (q_n + q_{n+1})}$$

and this difference therefore diminishes as n increases.

Proof.—By taking the difference, $\frac{p_n}{q_n} - \frac{a' p_{n+1} + p_n}{a' q_{n+1} + q_n}$. (163)

Also F is nearer the true value than any other fraction with a less denominator.

166 $F_n F_{n+1}$ is greater or less than F^2 according as F_n is greater or less than F_{n+1}.

General Theory of Continued Fractions.

167 First class of continued fraction.

$$F = \frac{b_1}{a_1 +}\;\frac{b_2}{a_2 +}\;\frac{b_3}{a_3 + \text{\&c.}}$$

Second class of continued fraction.

$$F = \frac{b_1}{a_1 -}\;\frac{b_2}{a_2 -}\;\frac{b_3}{a_3 - \text{\&c.}}$$

a_1, b_1, &c. are taken as positive quantities.

$\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$, &c. are termed *components* of the continued fraction. If the components be infinite in number, the continued fraction is said to be infinite.

Let the successive convergents be denoted by

$$\frac{p_1}{q_1} = \frac{b_1}{a_1}; \quad \frac{p_2}{q_2} = \frac{b_1}{a_1+}\;\frac{b_2}{a_2}; \quad \frac{p_3}{q_3} = \frac{b_1}{a_1+}\;\frac{b_2}{a_2+}\;\frac{b_3}{a_3}; \text{ and so on.}$$

168 The law of formation of the convergents is

For F,	For V,
$\begin{cases} p_n = a_n p_{n-1} + b_n p_{n-2} \\ q_n = a_n q_{n-1} + b_n q_{n-2} \end{cases}$	$\begin{cases} p_n = a_n p_{n-1} - b_n p_{n-2} \\ q_n = a_n q_{n-1} - b_n q_{n-2} \end{cases}$

[Proved by Induction.

The relation between the successive differences of the convergents is, by (168),

169 $$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \mp \frac{b_{n+1} q_{n-1}}{q_{n+1}} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right).$$

Take the $-$ sign for F, and the $+$ for V.

170 $$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 b_3 \dots b_n. \qquad (168)$$

171 The odd convergents for F, $\frac{p_1}{q_1}$, $\frac{p_3}{q_3}$, &c., continually decrease, and the even convergents, $\frac{p_2}{q_2}$, $\frac{p_4}{q_4}$, &c., continually increase. (167)

Every odd convergent is greater, and every even convergent is less, than all following convergents. (169)

172 Def.—If the difference between consecutive convergents diminishes without limit, the infinite continued fraction is said to be *definite*. If the same difference tends to a fixed value greater than zero, the infinite continued fraction is *indefinite*; the odd convergents tending to one value, and the even convergents to another.

173 F is definite if the ratio of every quotient to the next component is greater than a fixed quantity.

PROOF.—Apply (169) successively.

174 F is incommensurable when the components are all proper fractions and infinite in number.

PROOF.—Indirectly, and by (168).

175 If a be never less than $b+1$, the convergents of V are all positive proper fractions, increasing in magnitude, p_n and q_n also increasing with n. By (167) and (168).

176 If, in this case, V be infinite, it is also definite, being $=1$, if a always $=b+1$ while b is less than 1, (175); and being less than 1, if a is ever greater than $b+1$. By (180).

177 V is incommensurable when it is less than 1, and the components are all proper fractions and infinite in number.

180 If in the continued fraction V (167), we have $a_n = b_n + 1$ always; then, by (168),

$$p_n = b_1 + b_1 b_2 + b_1 b_2 b_3 + \dots \text{ to } n \text{ terms, and } q_n = p_n + 1.$$

---

181 If, in the continued fraction F, a_n and b_n are constant and equal, say, to a and b respectively; then p_n and q_n are respectively equal to the coefficients of x^{n-1} in the expansions

of $$\frac{b}{1-ax-bx^2} \quad \text{and} \quad \frac{a+bx}{1-ax-bx^2}.$$

PROOF.—p_n and q_n are the n^{th} terms of two recurring series. See (168) and (251).

---

182 *To convert a Series into a Continued Fraction.*

The series $$\frac{1}{u} + \frac{x}{u_1} + \frac{x^2}{u_2} + \dots + \frac{x^n}{u_n}$$

is equal to a continued fraction V (167), with $n+1$ components; the first, second, and $n+1^{\text{th}}$ components being

$$\frac{1}{u}, \quad \frac{u^2 x}{u_1 + ux}, \quad \dots \quad \frac{u_{n-1}^2 x}{u_n + u_{n-1} x}.$$

[Proved by Induction.

183 The series

$$\frac{1}{r}+\frac{x}{rr_1}+\frac{x^2}{rr_1r_2}+\ldots+\frac{x^n}{rr_1r_2\ldots r_n}$$

is equal to a continued fraction V (167), with $n+1$ components, the first, second, and $n+1^{\text{th}}$ components being

$$\frac{1}{r},\quad \frac{rx}{r_1+x},\quad \ldots\ldots\quad \frac{r_{n-1}x}{r_n+x}.$$ [Proved by Induction.

184 The sign of x may be changed in either of the statements in (182) or (183).

185 Also, if any of these series are convergent and infinite, the continued fractions become infinite.

186 *To find the value of a continued fraction with recurring quotients.*

Let the continued fraction be

$$x=\frac{b_1}{a_1+}\ldots+\frac{b_n}{a_n+y}\quad\text{where}\quad y=\frac{b_{n+1}}{a_{n+1}+}\ldots+\frac{b_{n+m}}{a_{n+m}+y}$$

so that there are m recurring quotients. Form the n^{th} convergent for x, and the m^{th} for y. Then, by substituting the complete quotients a_n+y for a_n, and $a_{n+m}+y$ for a_{n+m} in (168), two equations are obtained of the forms

$$x=\frac{Ay+B}{Cy+D}\quad\text{and}\quad y=\frac{Ey+F}{Gy+H},$$

from which, by eliminating y, a quadratic equation for determining x is obtained.

187 If

$$\frac{b_1}{a_1+}\ldots+\frac{b_n}{a_n+}$$

be a continued fraction, and

$$\frac{p_1}{q_1},\quad\ldots\ldots\quad\frac{p_n}{q_n}$$

the corresponding first n convergents; then $\frac{q_{n-1}}{q_n}$, developed by (168), produces the continued fraction

$$\frac{1}{a_n+}\,\frac{b_n}{a_{n-1}+}\,\frac{b_{n-1}}{a_{n-2}+}\dots+\frac{b_3}{a_2+}\,\frac{b_2}{a_1}$$

the quotients being the same but in reversed order.

INDETERMINATE EQUATIONS.

188 Given $$ax+by=c$$
free from fractions, and α, β integral values of x and y which satisfy the equation, the complete integral solution is given by

$$x=\alpha-bt$$
$$y=\beta+at$$

where t is any integer.

EXAMPLE.—Given $5x+3y=112$.

Then $x=20$, $y=4$ are values;

$$\therefore\quad \left.\begin{aligned} x &= 20-3t \\ y &= 4+5t \end{aligned}\right\}.$$

The values of x and y may be exhibited as under:

$t=$	-2	-1	0	1	2	3	4	5	6	7
$x=$	26	23	20	17	14	11	8	5	2	-1
$y=$	-6	-1	4	9	14	19	24	29	34	39

For solutions in positive integers t must lie between $\frac{20}{3}=6\frac{2}{3}$ and $-\frac{4}{5}$; that is, t must be 0, 1, 2, 3, 4, 5, or 6, giving 7 positive integral solutions.

189 If the equation be

$$ax-by=c$$

the solutions are given by

$$x=\alpha+bt$$
$$y=\beta+at.$$

EXAMPLE: $4x - 3y = 19.$

Here $x = 10$, $y = 7$ satisfy the equation;

$$\therefore \quad \left.\begin{aligned} x &= 10+3t \\ y &= 7+4t \end{aligned}\right\} \text{ furnish all the solutions.}$$

The simultaneous values of t, x, and y will be as follows:—

$t = -5$	-4	-3	-2	-1	0	1	2	3
$x = -5$	-2	1	4	7	10	13	16	19
$y = -13$	-9	-5	-1	3	7	11	15	19

The number of positive integral solutions is infinite, and the least positive integral values of x and y are given by the limiting value of t, viz.,

$$t > -\frac{10}{3} \quad \text{and} \quad t > -\frac{7}{4};$$

that is, t must be -1, 0, 1, 2, 3, or greater.

190 If two values, α and β, cannot readily be found by inspection, as, for example, in the equation

$$17x + 13y = 14900,$$

divide by the least coefficient, and equate the remaining fractions to t, an integer; thus

$$y + x + \frac{4x}{13} = 1146 + \frac{2}{13} \quad \text{.........................} (1).$$

$$\therefore \quad 4x - 2 = 13t.$$

Repeat the process; thus

$$x - \frac{2}{4} = 3t + \frac{t}{4},$$

$$\therefore \quad t + 2 = 4u.$$

Put $u = 1,$

$$\therefore \quad t = 2,$$

$$x = \frac{13t+2}{4} = 7 = \alpha;$$

and $y + x + t = 1146$, by (1),

$$\therefore \quad y = 1146 - 7 - 2 = 1137 = \beta.$$

The general solution will be

$$x = 7 - 13t,$$
$$y = 1137 + 17t,$$

Or, changing the sign of t for convenience,

$$x = 7 + 13t,$$
$$y = 1137 - 17t.$$

Here the number of solutions in positive integers is equal to the number of integers lying between $-\frac{7}{13}$ and $\frac{1137}{17}$;

or $-\frac{7}{13}$ and $66\frac{15}{17}$; that is, 67.

191 Otherwise.—Two values of x and y may be found in the following manner:—

Find the nearest converging fraction to $\frac{17}{13}$. [By (160).

This is $\frac{4}{3}$. By (164) we have

$$17\times 3-13\times 4=-1.$$

Multiply by 14900, and change the signs;

$$\therefore \quad 17(-44700)+13(59600)=14900;$$

which shews that we may take $\begin{cases}\alpha=-44700\\ \beta=\quad 59600\end{cases}$

and the general solution may be written

$$x=-44700+13t,$$
$$y=\quad 59600-17t.$$

This method has the disadvantage of producing high values of α and β.

192 The values of x and y, in positive integers, which satisfy the equation $ax\pm by=c$, form two Arithmetic Progressions, of which b and a are respectively the common differences. See examples (188) and (189).

193 Abbreviation of the method in (169).

Example: $11x-18y=63$.

Put $x=9z$, and divide by 9; then proceed as before.

194 *To obtain integral solutions of $ax+by+cz=d$.*

Write the equation thus

$$ax+by=d-cz.$$

Put successive integers for z, and solve for x, y in each case.

TO REDUCE A QUADRATIC SURD TO A CONTINUED FRACTION.

195 EXAMPLE:

$$\sqrt{29} = 5 + \sqrt{29} - 5 = 5 + \frac{4}{\sqrt{29}+5},$$

$$\frac{\sqrt{29}+5}{4} = 2 + \frac{\sqrt{29}-3}{4} = 2 + \frac{5}{\sqrt{29}+3},$$

$$\frac{\sqrt{29}+3}{5} = 1 + \frac{\sqrt{29}-2}{5} = 1 + \frac{5}{\sqrt{29}+2},$$

$$\frac{\sqrt{29}+2}{5} = 1 + \frac{\sqrt{29}-3}{5} = 1 + \frac{4}{\sqrt{29}+3},$$

$$\frac{\sqrt{29}+3}{4} = 2 + \frac{\sqrt{29}-5}{4} = 2 + \frac{1}{\sqrt{29}+5},$$

$$\sqrt{29}+5 = 10 + \sqrt{29}-5 = 10 + \frac{4}{\sqrt{29}+5}.$$

The quotients 5, 2, 1, 1, 2, 10 are the greatest integers contained in the quantities in the first column. The quotients now recur, and the surd $\sqrt{29}$ is equivalent to the continued fraction

$$5 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{10+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\&c.$$

The convergents to $\sqrt{29}$, formed as in (160), will be

$$\frac{5}{1},\quad \frac{11}{2},\quad \frac{16}{3},\quad \frac{27}{5},\quad \frac{70}{13},\quad \frac{727}{135},\quad \frac{1524}{283},\quad \frac{2251}{418},\quad \frac{3775}{701},\quad \frac{9801}{1820}.$$

196 Note that the last quotient 10 is the greatest and twice the first, that the *second* is the first of the recurring ones, and that the recurring quotients, excluding the last, consist of pairs of equal terms, quotients equi-distant from the first and last being equal. These properties are universal. (See 204–210).

To form high convergents rapidly.

197 Suppose m the number of recurring quotients, or any

multiple of that number, and let the m^{th} convergent to $\sqrt{Q}$ be represented by F_m; then the $2m^{\text{th}}$ convergent is given by the formula

$$F_{2m} = \frac{1}{2}\left\{F_m + \frac{Q}{F_m}\right\} \text{ by (203) and (210).}$$

198 For example, in approximating to $\sqrt{29}$ above, there are five recurring quotients. Take $m = 2\times 5 = 10$; therefore, by

$$F_{20} = \frac{1}{2}\left\{F_{10} + \frac{29}{F_{10}}\right\},$$

$$F_{10} = \frac{9801}{1820}, \text{ the } 10^{\text{th}} \text{ convergent.}$$

Therefore $$F_{20} = \left\{\frac{9801}{1820} + 29\times\frac{1820}{9801}\right\} = \frac{192119201}{35675640}$$

the 20^{th} convergent to $\sqrt{29}$; and the labour of calculating the intervening convergents is saved.

GENERAL THEORY.

199 The process of (174) may be exhibited as follows :—

$$\frac{\sqrt{Q}+c_1}{r_1} = a_1 + \frac{r_2}{\sqrt{Q}+c_2}$$

$$\frac{\sqrt{Q}+c_2}{r_2} = a_2 + \frac{r_3}{\sqrt{Q}+c_3}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{\sqrt{Q}+c_n}{r_n} = a_n + \frac{r_{n+1}}{\sqrt{Q}+c_{n+1}}.$$

200 Then $$\sqrt{Q} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+}\ \&c.$$

The quotients a_1, a_2, a_3, &c. are the integral parts of the fractions on the left.

201 The equations connecting the remaining quantities are

$$c_1 = 0 \qquad\qquad r_1 = 1$$

$$c_2 = a_1 r_1 - c_1 \qquad\qquad r_2 = \frac{Q - c_2^2}{r_1}$$

$$c_3 = a_2 r_2 - c_2 \qquad\qquad r_3 = \frac{Q - c_3^2}{r_2}$$

$$\dots \quad \dots \quad \dots \qquad\qquad \dots \quad \dots \quad \dots$$

$$c_n = a_{n-1} r_{n-1} - c_{n-1} \qquad\qquad r_n = \frac{Q - c_n^2}{r_{n-1}}$$

The n^{th} convergent to $\sqrt{Q}$ will be

202 $$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$ [By Induction.

The true value of $\sqrt{Q}$ is what this becomes when we substitute for a_n the complete quotient $\dfrac{\sqrt{Q}+c_n}{r_n}$, of which a_n is only the integral part. This gives

203 $$\sqrt{Q} = \frac{(\sqrt{Q}+c_n)\, p_{n-1} + r_n p_{n-2}}{(\sqrt{Q}+c_n)\, q_{n-1} + r_n q_{n-2}}.$$

By the relations (199) to (203) the following theorems are demonstrated :—

204 All the quantities a, r, and c are positive integers.

205 The greatest c is c_2, and $c_2 = a_1$.

206 No a or r can be greater than $2a_1$.

207 If $r_n = 1$, then $c_n = a_1$.

208 For all values of n greater than 1, $a - c_n$ is $< r_n$.

209 The number of quotients cannot be greater than $2a_1^2$. The last quotient is $2a_1$, and after that the terms repeat. The first complete quotient that is repeated is $\dfrac{\sqrt{Q}+c_2}{r_2}$, and a_2, r_2, c_2 commence each cycle of repeated terms.

210 Let a_m, r_m, c_m be the last terms of the first cycle; then a_{m-1}, r_{m-1}, c_{m-1} are respectively equal to a_2, r_2, c_2; a_{m-2}, r_{m-2}, c_{m-2} are equal to a_3, r_3, c_3, and so on. [By (187).

EQUATIONS.

Special Cases in the Solution of Simultaneous Equations.

211 First, with two unknown quantities.

$$\left.\begin{aligned} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{aligned}\right\}. \quad x = \frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}, \quad y = \frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}.$$

If the denominators vanish, we have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}, \text{ and } x = \infty,\ y = \infty;$$

unless at the same time the numerators vanish, for then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}; \quad x = \frac{0}{0}; \quad y = \frac{0}{0};$$

and the equations are not *independent*, one being produced by multiplying the other by some constant.

212 Next, with three unknown quantities. See (60) for the equations.

If d_1, d_2, d_3 all vanish, divide each equation by z, and we have three equations for finding the two ratios $\frac{x}{z}$ and $\frac{y}{z}$, two only of which equations are necessary, any one being deducible from the other two if the three be consistent.

213 *To solve simultaneous equations by Indeterminate Multipliers.*

Ex.—Take the equations

$$\begin{aligned} x+2y+\ 3z+4w &= 27, \\ 3x+5y+\ 7z+\ w &= 48, \\ 5x+8y+10z-2w &= 65, \\ 7x+6y+\ 5z+4w &= 53. \end{aligned}$$

Multiply the first by A, the second by B, the third by C, leaving one equation unmultiplied; and then add the results.

Thus
$$(A+3B+5C+7)\,x+(2A+5B+8C+6)\,y$$
$$+(3A+7B+10C+5)\,z+(4A+B-2C+4)\,w$$
$$=27A+48B+65C+53.$$

To determine either of the unknowns, for instance x, equate the coefficients of the other three separately to zero, and from the three equations find A, B, C. Then

$$x=\frac{27A+48B+65C+53}{A+3B+5C+7}.$$

MISCELLANEOUS EQUATIONS AND SOLUTIONS.

214 $$x^6 \pm 1 = 0.$$

Divide by x^3, and throw into factors, by (2) or (3). See also (480).

215 $$x^3-7x-6=0.$$

$x=-1$ is a root, by inspection; therefore $x+1$ is a factor. Divide by $x+1$, and solve the resulting quadratic.

216 $$x^3+16x=455.$$

$$x^4+16x^2=455x=65\times 7x,$$
$$x^4+65x^2+\left(\frac{65}{2}\right)^2=49x^2+65\times 7x+\left(\frac{65}{2}\right)^2,$$
$$x^2+\frac{65}{2}=7x+\frac{65}{2},$$
$$x^2=7x; \qquad \therefore\quad x=7.$$

Rule.—Divide the absolute term (here 455) into two factors, if possible, such that one of them, minus the square of the other, equals the coefficient of x. See (483) for general solution of a cubic equation.

217 $x^4 - y^4 = 14560, \quad x - y = 8.$

Put $x = z + v$ and $y = z - v.$

Eliminate v, and obtain a cubic in z, which solve as in (216).

218 $x^5 - y^5 = 3093, \quad x - y = 3.$

Divide the first equation by the second, and subtract from the result the fourth power of $x - y$. Eliminate $(x^2 + y^2)$, and obtain a quadratic in xy.

219 *On forming Symmetrical Expressions.*

Take, for example, the equation

$$(y - c)(z - b) = a^2.$$

To form the remaining equations symmetrical with this, write the corresponding letters in *vertical* columns, observing the circular order in which a is followed by b, b by c, and c by a. So with x, y, and z. Thus the equations become

$$(y - c)(z - b) = a^2,$$
$$(z - a)(x - c) = b^2,$$
$$(x - b)(y - a) = c^2.$$

To solve these equations, substitute

$$x = b + c + x', \quad y = c + a + y', \quad z = a + b + z';$$

and, multiplying out, and eliminating y and z, we obtain

$$x = \frac{bc(b + c) - a(b^2 + c^2)}{bc - ca - ab},$$

and therefore, by symmetry, the values of y and z, by the rule just given.

220 $y^2 + z^2 + yz = a^2$ .... (1),

$z^2 + x^2 + zx = b^2$ .... (2),

$x^2 + y^2 + xy = c^2$ .... (3);

$\therefore \quad 3(yz + zx + xy)^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$ ...... (4).

Now add (1), (2), and (3), and we obtain

$$2(x+y+z)^2-3(yz+zx+xy) = a^2+b^2+c^2 \quad \ldots\ldots \quad (5).$$

From (4) and (5), $(x+y+z)$ is obtained, and then (1), (2), and (3) are readily solved.

221

$$x^2-yz = a^2 \quad \ldots\ldots \quad (1),$$

$$y^2-zx = b^2 \quad \ldots\ldots \quad (2),$$

$$z^2-xy = c^2 \quad \ldots\ldots \quad (3).$$

Multiply (2) by (3), and subtract the square of (1).

Result $\quad x(3xyz-x^3-y^3-z^3) = b^2c^2-a^4,$

$$\therefore \quad \frac{x}{b^2c^2-a^4} = \frac{y}{c^2a^2-b^4} = \frac{z}{a^2b^2-c^4} = \lambda \quad \ldots\ldots \quad (4).$$

Obtain λ^2 by proportion as a fraction with numerator

$$= x^2-yz = a^2.$$

222

$$x = cy+bz \quad \ldots\ldots \quad (1),$$

$$y = az+cx \quad \ldots\ldots \quad (2),$$

$$z = bx+ay \quad \ldots\ldots \quad (3).$$

Eliminate a between (2) and (3), and substitute the value of x from equation (1).

Result $\quad \dfrac{y^2}{1-b^2} = \dfrac{z^2}{1-c^2} = \dfrac{x^2}{1-a^2}.$

IMAGINARY EXPRESSIONS.

223 The following are conventions:—

That $\sqrt{(-a^2)}$ is equivalent to $a\sqrt{(-1)}$; that $a\sqrt{(-1)}$ vanishes when a vanishes; that the symbol $a\sqrt{(-1)}$ is subject to the ordinary rules of Algebra. $\sqrt{(-1)}$ is denoted by i.

224 If $\alpha+i\beta=\gamma+i\delta$; then $\alpha=\gamma$ and $\beta=\delta$.

225 $\alpha+i\beta$ and $\alpha-i\beta$ are conjugate expressions; their product $=\alpha^2+\beta^2$.

226 The sum and product of two conjugate expressions are both real, but their difference is imaginary.

227 The modulus is $+\sqrt{\alpha^2+\beta^2}$.

228 If the modulus vanishes, α and β must vanish.

229 If two imaginary expressions are equal, their moduli are equal, by (224).

230 The modulus of the product of two imaginary expressions is equal to the product of their moduli.

231 Also the modulus of the quotient is equal to the quotient of their moduli.

METHOD OF INDETERMINATE COEFFICIENTS.

232 If $A+Bx+Cx^2+\ldots=A'+B'x+C'x^2+\ldots$ be an equation which holds for all values of x, the coefficients A, B, &c. not involving x, then $A=A'$, $B=B'$, $C=C'$, &c.; that is, the coefficients of like powers of x must be equal. Proved by putting $x=0$, and dividing by x alternately. See (234) for an example.

233 METHOD OF PROOF BY INDUCTION.

Ex.—To prove that

$$1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Assume $$1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6};$$

$$\therefore \quad 1+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6}+(n+1)^2$$

$$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\{n(2n+1)+6(n+1)\}}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{n'(n'+1)(2n'+1)}{6},$$

where n' is written for $n+1$;

$$\therefore \quad 1+2^2+3^2+\ldots+n'^2 = \frac{n'(n'+1)(2n'+1)}{6}.$$

It is thus proved that *if the formula be true for n it is also true for n + 1.*

But the formula *is* true when $n = 2$ or 3, as may be shewn by actual trial; therefore it is true when $n = 4$; therefore also when $n = 5$, and so on; therefore universally true.

---

234 Ex.—The same theorem proved by the method of Indeterminate coefficients.

Assume

$$1+2^2+3^2+\ldots+n^2 = A+Bn+Cn^2+Dn^3+\&c.;$$

$$\therefore \; 1+2^2+3^2+\ldots+n^2+(n+1)^2 = A+B(n+1)+C(n+1)^2+D(n+1)^3+\&c.;$$

therefore, by subtraction,

$$n^2+2n+1 = B+C(2n+1)+D(3n^2+3n+1),$$

writing no terms in this equation which contain higher powers of n than the highest which occurs on the left-hand side, for the coefficients of such terms may be shewn to be separately equal to zero.

Now equate the coefficients of like powers of n; thus

$$3D = 1, \qquad \therefore \; D = \frac{1}{3};$$

$$2C+3D = 2, \qquad \therefore \; C = \frac{1}{2}, \quad \text{and} \quad A = 0;$$

$$B+C+D = 1, \qquad \therefore \; B = \frac{1}{6};$$

therefore the sum of the series is equal to

$$\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3} = \frac{n(n+1)(2n+1)}{6}.$$

PARTIAL FRACTIONS.

In the resolution of a fraction into partial fractions four cases present themselves, which are illustrated in the following examples.

235 First.—When there are no repeated factors in the denominator of the given fraction.

Ex.—To resolve $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$ into partial fractions.

Assume $$\frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3};$$

$$\therefore \quad 3x-2 = A(x-2)(x-3) + B(x-3)(x-1) + C(x-1)(x-2).$$

Since A, B, and C do not contain x, and this equation is true for all values of x, put $x=1$; then

$$3-2 = A(1-2)(1-3), \text{ from which } A = \frac{1}{2}.$$

Similarly, if x be put $= 2$, we have

$$6-2 = B(2-3)(2-1); \qquad \therefore \quad B = -4;$$

and, putting $x = 3$,

$$9-2 = C(3-1)(3-2); \qquad \therefore \quad C = \frac{7}{2}.$$

Hence $$\frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{7}{2(x-3)}.$$

236 Secondly.—When there is a repeated factor.

Ex.—Resolve into partial fractions $\dfrac{7x^3-10x^2+6x}{(x-1)^3(x+2)}$.

Assume $$\frac{7x^3-10x^2+6x}{(x-1)^3(x+2)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+2}.$$

These forms are necessary and sufficient. Multiplying up, we have

$$7x^3-10x^2+6x = A(x+2) + B(x-1)(x+2) + C(x-1)^2(x+2) + D(x-1)^3 \quad \text{.........} (1).$$

Make $x=1$; $\quad \therefore \quad 7-10+6 = A(1+2); \quad \therefore \quad A = 1.$

Substitute this value of A in (1); thus

$$7x^3-10x^2+5x-2 = B(x-1)(x+2) + C(x-1)^2(x+2) + D(x-1)^3.$$

Divide by $x-1$; thus

$$7x^2-3x+2 = B(x+2) + C(x-1)(x+2) + D(x-1)^2 \quad \text{.........} (2).$$

Make $x=1$ again, $\quad 7-3+2 = B(1+2); \quad \therefore \quad B = 2.$

Substitute this value of B in (2), and we have

$$7x^2-5x-2 = C(x-1)(x+2) + D(x-1)^2.$$

Divide by $x-1$, $\quad 7x+2 = C(x+2) + D(x-1) \quad \text{........................} (3).$

Put $x=1$ a third time, $\quad 7+2 = C(1+2); \quad \therefore \quad C = 3.$

Lastly, make $x=-2$ in (3),

$$-14+2 = D(-2-1); \quad \therefore \quad D = 4.$$

Result $$\frac{1}{(x-1)^3}+\frac{2}{(x-1)^2}+\frac{3}{x-1}+\frac{4}{x+2}.$$

237 Thirdly.—When there is a quadratic factor of imaginary roots not repeated.

Ex.—Resolve $\dfrac{1}{(x^2+1)(x^2+x+1)}$ into partial fractions.

Here we must assume

$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+x+1};$$

x^2+1 and x^2+x+1 have no real factors, and are therefore retained as denominators. The requisite form of the numerators is seen by adding together two simple fractions, such as $\dfrac{a}{x+b}+\dfrac{c}{x+d}$.

Multiplying up, we have the equation

$$1 = (Ax+B)(x^2+x+1)+(Cx+D)(x^2+1) \quad\text{..............} \quad (1).$$

Let $$x^2+1=0; \quad \therefore \quad x^2=-1.$$

Substitute this value of x^2 in (1) repeatedly; thus

$$1 = (Ax+B)x = Ax^2+Bx = -A+Bx;$$

or $$Bx-A-1 = 0.$$

Equate coefficients to zero; $\therefore \quad B = 0,$

$$A = -1.$$

Again, let $$x^2+x+1 = 0;$$

$$\therefore \quad x^2 = -x-1.$$

Substitute this value of x^2 repeatedly in (1); thus

$$1 = (Cx+D)(-x) = -Cx^2-Dx = Cx+C-Dx;$$

or $$(C-D)x+C-1 = 0.$$

Equate coefficients to zero; thus $C = 1,$

$$D = 1.$$

Hence $$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{x+1}{x^2+x+1}-\frac{x}{x^2+1}.$$

238 Fourthly.—When there is a repeated quadratic factor of imaginary roots.

Ex.—Resolve $\dfrac{40x-103}{(x+1)^2(x^2-4x+8)^3}$ into partial fractions.

Assume

$$\frac{40x-103}{(x+1)^2(x^2-4x+8)^3} = \frac{Ax+B}{(x^2-4x+8)^3} + \frac{Cx+D}{(x^2-4x+8)^2} + \frac{Ex+F}{x^2-4x+8} + \frac{G}{(x+1)^2} + \frac{H}{x+1};$$

$\therefore$

$$40x-103 = \{(Ax+B)+(Cx+D)(x^2-4x+8)+(Ex+F)(x^2-4x+8)^2\}(x+1)^2 + \{G+H(x+1)\}(x^2-4x+8)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the first place, to determine A and B, equate x^2-4x+8 to zero; thus $x^2 = 4x-8$.

Substitute this value of x^2 repeatedly in (1), as in the previous example, until the first power of x alone remains. The resulting equation is

$$40x-103 = (17A+6B)\,x-48A-7B.$$

Equating coefficients, we obtain two equations

$$\left.\begin{aligned} 17A+6B &= 40 \\ 48A+7B &= 103 \end{aligned}\right\}, \quad \text{from which} \quad \begin{aligned} A &= 2 \\ B &= 1. \end{aligned}$$

Next, to determine C and D, substitute these values of A and B in (1); the equation will then be divisible by x^2-4x+8. Divide, and the resulting equation is

$$0 = 2x+13+\{Cx+D+(Ex+F)(x^2-4x+8)\}(x+1)^2 + \{G+H(x+1)\}(x^2-4x+8)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equate x^2-4x+8 again to zero, and proceed exactly as before, when finding A and B.

Next, to determine E and F, substitute the values of C and D, last found in equation (2); divide, and proceed as before.

Lastly, G and H are determined by equating $x+1$ to zero successively, as in Example 2.

CONVERGENCY AND DIVERGENCY OF SERIES.

239 Let $a_1+a_2+a_3+$&c. be a series, and a_n, a_{n+1} any two consecutive terms. The following tests of convergency may be applied. The series will converge, if, after any fixed term—

(i.) The terms decrease and are alternately positive and negative.

(ii.) Or if $\frac{a_n}{a_{n+1}}$ is always *greater* than some quantity greater than unity.

(iii.) Or if $\frac{a_n}{a_{n+1}}$ is never less than the corresponding ratio in a known converging series.

(iv.) Or if $\left(\frac{na_n}{a_{n+1}} - n\right)$ is always *greater* than some quantity greater than unity. [By 244 and iii.

(v.) Or if $\left(\frac{na_n}{a_{n+1}} - n - 1\right) \log n$ is always *greater* than some quantity greater than unity.

240 The conditions of divergency are obviously the converse of rules (i.) to (v.).

241 The series $a_1 + a_2x + a_3x^2 + \&c.$ converges, if $\frac{a_{n+1}}{a_n}$ is always less than some quantity p, and x less than $\frac{1}{p}$.

[By 239 (ii.)

242 To make the sum of the last series less than an assigned quantity p, make x less than $\frac{p}{p+k}$, k being the greatest coefficient.

General Theorem.

243 If $\phi(x)$ be positive for all positive integral values of x, and continually diminish as x increases, and if m be any positive integer, then the two series

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \ldots\ldots$$

$$\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \ldots\ldots$$

are either both convergent or divergent.

244 Application of this theorem. To ascertain whether the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots$$

is divergent or convergent when p is greater than unity.

Taking $m=2$, the second series in (243) becomes

$$1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\&c.\ldots\ldots,$$

a geometrical progression which converges; therefore the given series converges.

245 The series of which $\dfrac{1}{n(\log n)^p}$ is the general term is convergent if p be greater than unity, and divergent if p be not greater than unity. [By (243), (244).

246 The series of which the general term is

$$\frac{1}{n\lambda(n)\,\lambda^2(n)\ldots\ldots\lambda^r(n)\,\{\lambda^{r+1}(n)\}^p},$$

where $\lambda(n)$ signifies $\log n$, $\lambda^2(n)$ signifies $\log\{\log(n)\}$, and so on, is convergent if p be greater than unity, and divergent if p be not greater than unity. [By Induction, and by (243).

247 The series $a_1+a_2+\&c.$ is convergent if

$$na_n\log(n)\log^2(n)\ldots\ldots\log^r(n)\{\log_{r+1}(n)\}^p$$

is always finite for a value of p greater than unity; $\log^2(n)$ here signifying $\log(\log n)$, and so on.

[See Todhunter's *Algebra*, or Boole's *Finite Differences*.

EXPANSION OF A FRACTION.

248 A fractional expression such as $\dfrac{4x-10x^2}{1-6x+11x^2-6x^3}$ may be expanded in ascending powers of x in three different ways.

First, by dividing the numerator by the denominator in the ordinary way, or by Synthetic Division, as shewn in (28).

Secondly, by the method of Indeterminate Coefficients (232).

Thirdly, by Partial Fractions and the Binomial Theorem.

To expand by the method of Indeterminate Coefficients, proceed as follows:—

Assume $\dfrac{4x-10x^2}{1-6x+11x^2-6x^3} = A+Bx+Cx^2+Dx^3+Ex^4+\&c.$

$$\begin{aligned}\therefore\quad 4x-10x^2 = A+\ & Bx+\ Cx^2+\ Dx^3+\ Ex^4+\ Fx^5+\ldots\\ & -6Ax-\ 6Bx^2-\ 6Cx^3-\ 6Dx^4-\ 6Ex^5-\ldots\\ & \qquad +11Ax^2+11Bx^3+11Cx^4+11Dx^5+\ldots\\ & \qquad\qquad -\ 6Ax^3-\ 6Bx^4-\ 6Cx^5-\ldots\end{aligned}$$

Equate coefficients of like powers of x, thus

$$\begin{array}{rcll} A &=& 0, & \\ B-\ 6A &=& 4, & \therefore\ B=\ 4;\\ C-\ 6B+11A &=& -10, & \therefore\ C=\ 14;\\ D-6C+11B-\ 6A &=& 0, & \therefore\ D=\ 40;\\ E-6D+11C-\ 6B &=& 0, & \therefore\ E=110;\\ F-6E+11D-\ 6C &=& 0, & \therefore\ F=304;\\ \ldots & \ldots & \ldots & \ldots \end{array}$$

The formation of the same coefficients by synthetic division is now exhibited, in order that the connexion between the two processes may be clearly seen.

The division of $4x-10x^2$ by $1-6x+11x^2-6x^3$ is as follows:—

$$\begin{array}{r|l} & 0+4-10 \\ +\ 6 & \qquad 24+84+240+660 \\ -11 & \qquad\quad -44-154-440-1210 \\ +\ 6 & \qquad\qquad +\ 24+\ 84+\ 240+660 \\ \hline & 0+4+14+40+110+304+\ \ldots\ldots\ldots\ldots \\ & A\quad B\quad C\quad D\quad E\quad F \end{array}$$

If we stop at the term $110x^4$, then the undivided remainder will be $304x^5-970x^6+660x^7$, and the complete result will be

$$4x+14x^2+40x^3+110x^4+\frac{304x^5-970x^6+660x^7}{1-6x+11x^2-6x^3}.$$

249 Here the concluding fraction may be regarded as the sum to infinity after four terms of the series, just as the original expression is considered to be the sum to infinity of the whole series.

250 If the general term be required, the method of expansion by partial fractions must be adopted. See (257), where the general term of the foregoing series is obtained.

RECURRING SERIES.

$a_0+a_1x+a_2x^2+a_3x^3+$ &c. is a recurring series if the coefficients are connected by the relation

251 $$a_n = p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_m a_{n-m}.$$

The Scale of Relation is

252 $$1 - p_1 x - p_2 x^2 - \dots - p_m x^m.$$

The sum of n terms of the series is equal to

253 [The first m terms
$-p_1x$ (first $m-1$ terms + the last term)
$-p_2x^2$ (first $m-2$ terms + the last 2 terms)
$-p_3x^3$ (first $m-3$ terms + the last 3 terms)
... ...
$-p_{m-1}x^{m-1}$ (first term + the last $m-1$ terms)
$-p_mx^m$ (the last m terms)] $\div$ $[1-p_1x-p_2x^2-\dots-p_mx^m]$.

254 If the series converges, and the sum to infinity is required, omit all "the last terms" from the formula.

255 Example.—Required the Scale of Relation, the general term, and the apparent sum to infinity, of the series

$$4x+14x^2+40x^3+110x^4+304x^5-854x^6+\dots.$$

Observe that six arbitrary terms given are sufficient to determine a Scale of Relation of the form $1-px-qx^2-rx^3$, involving three constants p, q, r, for, by (251), we can write three equations to determine these constants; namely,

$$\left.\begin{aligned} 110 &= 40p+14q+4r \\ 304 &= 110p+40q+14r \\ 854 &= 304p+110q+40r \end{aligned}\right\}.$$

The solution gives $p=6$, $q=-11$, $r=6$.

Hence the Scale of Relation is $1-6x+11x^2-6x^3$.

The sum of the series without limit will be found from (254), by putting $p_1=6$, $p_2=-11$, $p_3=6$, $m=3$.

The first three terms	=	$4x+14x^2+40x^3$
$-6\times$ the first two terms	=	$-24x^2-84x^3$
$+11x\times$ the first term	=	$+44x^3$
		$4x-10x^2$

$$\therefore \qquad S = \frac{4x-10x^2}{1-6x+11x^2-6x^3};$$

the meaning of which is that, if this fraction be expanded in ascending powers of x, the first six terms will be those given in the question.

256 To obtain more terms of the series, we may use the Scale of Relation; thus the 7th term will be

$$(6 \times 854 - 11 \times 304 + 6 \times 110)\, x^7 = 2440x^7.$$

257 To find the general term, S must be decomposed into partial fractions; thus, by the method of (235),

$$\frac{4x-10x^2}{1-6x+11x^2-6x^3} = \frac{1}{1-3x} + \frac{2}{1-2x} - \frac{3}{1-x}.$$

By the Binomial Theorem (128),

$$\frac{1}{1-3x} = \quad 1+3x+3^2x^2+\ldots\ldots+3^nx^n,$$

$$\frac{2}{1-2x} = \quad 2+2^2x+2^3x^2+\ldots\ldots+2^{n+1}x^n,$$

$$-\frac{3}{1-x} = -3-3x-3x^2-\ldots\ldots-3x^n.$$

Hence the general term involving x^n is

$$(3^n+2^{n+1}-3)\, x^n.$$

And by this formula we can write the "last terms" required in (253), and so obtain the sum of any finite number of terms of the given series. Also, by the same formula we can calculate the successive terms at the beginning of the series. In the present case this mode will be more expeditious than that of employing the Scale of Relation.

258 If, in decomposing S into partial fractions for the sake of obtaining the general term, a quadratic factor with imaginary roots should occur as a denominator, the same method must be pursued, for the imaginary quantities will disappear in the final result. In this case, however, it is more convenient to employ a general formula. Suppose the fraction which gives rise to the imaginary roots to be

$$\frac{L+Mx}{a+bx+x^2} = \frac{L+Mx}{(p-x)(q-x)},$$

p and q being the imaginary roots of $a+bx+x^2=0$.

Suppose $\qquad p = \alpha+i\beta,$

$\qquad q = \alpha-i\beta$, where $i = \sqrt{-1}$.

If, now, the above fraction be resolved into two partial fractions in the ordinary way, and if these fractions be expanded separately by the Binomial Theorem, and that part of the general term furnished by these two expansions written out, still retaining p and q, and if the imaginary values of p and q be then substituted, it will be found that the factor will disappear, and that the result may be enunciated as follows.

259 The coefficient of x^{n-1} in the expansion of

$$\frac{L+Mx}{(\alpha^2+\beta^2)-2\alpha x+x^2}$$

rrata

will be

$$\frac{L}{\beta(\alpha^2+\beta^2)}\left\{n\alpha^{n-1}\beta-C(n,3)\,\alpha^{n-3}\beta^3+C(n,5)\,\alpha^{n-5}\beta^5-\dots\right\}$$

$$+\frac{M}{\beta(\alpha^2+\beta^2)^{n-1}}\left\{(n-1)\,\alpha^{n-2}\beta-C(n-1,3)\,\alpha^{n-4}\beta^3+C(n-1,5)\,\alpha^{n-6}\beta^5-\dots\right\}.$$

260 With the aid of the known expansion of $\sin n\theta$ in Trigonometry, this formula for the n^{th} term may be reduced to

$$\sqrt{\frac{(L+M\alpha)^2+M^2\beta^2}{\beta^2(\alpha^2+\beta^2)^n}}\,.\sin(n\theta-\phi),$$

in which $\quad \theta=\tan^{-1}\dfrac{\beta}{\alpha}, \qquad \phi=\tan^{-1}\dfrac{M\beta}{L+M\alpha}.$

If n be not greater than 100, $\sin(n\theta-\phi)$ may be obtained from the tables correct to about six places of decimals, and accordingly the n^{th} term of the expansion may be found with corresponding accuracy. As an example, the 100^{th} term in the expansion of $\dfrac{1+x}{5-2x+x^2}$ is readily found by this method to be $\dfrac{41824}{10^{41}}\,x^{99}$.

To determine whether a given Series is recurring or not.

261 If certain first terms only of the series be given, a scale of relation may be found which shall produce a recurring

series whose first terms are those given. The method is exemplified in (255). The number of unknown coefficients p, q, r, &c. to be assumed for the scale of relation must be equal to half the number of the given terms of the series, if that number be even. If the number of given terms be odd, it may be made even by prefixing zero for the first term of the series.

262 Since this method may, however, produce zero values for one or more of the last coefficients in the scale of relation, it may be advisable in practice to determine a scale from the first two terms of the series, and if that scale does not produce the following terms, we may try a scale determined from the first four terms, and so on until the true scale is arrived at.

If an indefinite number of terms of the series be given, we may find whether it is recurring or not by a rule of Lagrange's.

263 Let the series be

$$S = A + Bx + Cx^2 + Dx^3 + \&c.$$

Divide unity by S as far as two terms of the quotient, which will be of the form $p + qx$, and write the remainder in the form $S'x^2$, S' being another indefinite series of the same form as S.

Next, divide S by S' as far as two terms of the quotient, and write the remainder in the form $S''x^2$.

Again, divide S' by S'', and proceed as before, and repeat this process until there is no remainder after one of the divisions. The series will then be proved to be a recurring series, and the order of the series, that is, the degree of the scale of relation, will be the same as the number of divisions which have been effected in the process.

Example.—To determine whether the series 1, 3, 6, 10, 15, 21, 28, 36, 45, ... is recurring or not.

Introducing x, we may write

$$S = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + 45x^8 \dots .$$

Then we shall have $\frac{S}{1} = 1 - 3x + \dots$ with a remainder

$$3x^2 + 8x^3 + 15x^4 + 24x^5 + 35x^6 + \&c.$$

Therefore $\quad S' = 3 + 8x + 15x^2 + 24x^3 + 35x^4 + \&c.,$

$$\frac{S}{S'} = \frac{1}{3} + \frac{x}{9}$$

with a remainder $\frac{1}{9}(x^2+3x^3+6x^4+10x^5+\&c. \ldots)$.

Therefore we may take $S'' = 1+3x+6x^2+10x^3+\&c.$

Lastly $\frac{S'}{S''} = 3-x$ without any remainder.

Consequently the series is a recurring series of the third order. It is, in fact, the expansion of $\frac{1}{1-3x+3x^2-x^3}$.

SUMMATION OF SERIES BY THE METHOD OF DIFFERENCES.

264 Rule.—*Form successive series of differences until a series of equal differences is obtained.* Let a, b, c, d, &c. be the first terms of the several series; then the n^{th} term of the given series is

265 $$a+(n-1)\,b+\frac{(n-1)(n-2)}{1.2}\,c+\frac{(n-1)(n-2)(n-3)}{1.2.3}\,d+$$

The sum of n terms

266 $$= na+\frac{n(n-1)}{1.2}\,b+\frac{n(n-1)(n-2)}{1.2.3}\,c+\&c.$$

Proved by Induction.

Example:

$a \ldots 1+\ 5+15+35+70+126+\ldots$
$b \ldots 4+10+20+35+56+\ldots$
$c \ldots 6+10+15+21+\ldots$
$d \ldots 4+\ 5+\ 6+\ldots$
$e \ldots 1+\ 1+\ldots$

The 100^{th} term of the first series

$$= 1+99.4+\frac{99.98}{1.2}6+\frac{99.98.97}{1.2.3}4+\frac{99.98.97.96}{1.2.3.4}.$$

The sum of 100 terms

$$= 100+\frac{100.99}{1.2}4+\frac{100.99.98}{1.2.3}6+\frac{100.99.98.97}{1.2.3.4}4+\frac{100.99.98.97.96}{1.2.3.4.5}.$$

267 To interpolate a term between two terms of a series by the method of differences.

Ex.—Given log 71, log 72, log 73, log 74, it is required to find log 72·54. Form the series of differences from the given logarithms, as in (266),

	log 71	log 72	log 73	log 74
a ...	1·8512583	1·8573325	1·8633229	1·8692317
b ...	·0060742	·0059904	·0059088	
c ...	−·0000838	−·0000816		
d ...	−·0000022	considered to vanish.		

Log 72·54 must be regarded as an interpolated term, the number of its place being 2·54.

Therefore put 2·54 for n in formula (265).

Result log 72·54 = 1·8605777.

DIRECT FACTORIAL SERIES.

268 Ex.: $5.7.9+7.9.11+9.11.13+11.13.15+\ldots$

d = common difference of factors,

m = number of factors in each term,

n = number of terms,

a = first factor of first term $-d$.

$$n^{\text{th}} \text{ term} = (a+nd)(a+\overline{n+1}\,d) \ldots\ldots (a+\overline{n+m-1}\,d).$$

269 To find the sum of n terms.

Rule.—*From the last term with the next highest factor take the first term with the next lowest factor, and divide by* $(m+1)\,d$.

Proof.—By Induction.

Thus the sum of 4 terms of the above series will be, putting $d=2$, $m=3$, $n=4$, $a=3$,

$$S = \frac{11.13.15.17-3.5.7.9}{(3+1)\,2}.$$

Proved either by Induction, or by the method of Indeterminate Coefficients.

INVERSE FACTORIAL SERIES.

270 Ex.: $\frac{1}{5.7.9}+\frac{1}{7.9.11}+\frac{1}{9.11.13}+\frac{1}{11.13.15}+\ldots$

Defining d, m, n, a as before, the

$$n^{\text{th}} \text{ term} = \frac{1}{(a+nd)\,(a+\overline{n+1}\,d)\ldots(a+\overline{n+m-1}\,d)}.$$

271 To find the sum of n terms. Rule.—*From the first term wanting its last factor take the last term wanting its first factor, and divide by* $(m-1)\,d$.

Thus the sum of 4 terms of the above series will be, putting $d=2$, $m=3$, $n=4$, $a=3$,

$$\frac{\frac{1}{5.7}-\frac{1}{13.15}}{(3-1)\,2}.$$

Proof.—By Induction, or by decomposing the terms, as in the following example.

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272 Ex.: To sum the same series by decomposing the terms into partial fractions. Take the general term in the simple form

$$\frac{2}{(r-2)\,r\,(r+2)}.$$

Resolve this into the three fractions

$$\frac{1}{8\,(r-2)}-\frac{1}{4r}+\frac{1}{8\,(r+2)} \text{ by (235).}$$

Substitute 7, 9, 11, &c. successively for r, and the given series has for its equivalent the three series

$$\frac{1}{8}\left\{\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\ \ldots\ldots\ldots\ldots\ +\frac{1}{2n+3}\right\}$$
$$+\frac{1}{8}\left\{-\frac{2}{7}-\frac{2}{9}-\frac{2}{11}-\frac{2}{13}-\ldots\ldots\ -\frac{2}{2n+3}-\frac{2}{2n+5}\right\}$$
$$+\frac{1}{8}\left\{\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\ldots\ldots+\frac{1}{2n+3}+\frac{1}{2n+5}+\frac{1}{2n+7}\right\},$$

and the sum of n terms is seen, by inspection, to be

$$\frac{1}{8}\left\{\frac{1}{5}-\frac{1}{7}-\frac{1}{2n+5}+\frac{1}{2n+7}\right\}=\frac{1}{4}\left\{\frac{1}{5.7}-\frac{1}{(2n+5)\,(2n+7)}\right\},$$

a result obtained at once by the rule in (271), taking $\frac{1}{5.7.9}$ for the first term, and $\frac{1}{(2n+3)\,(2n+5)\,(2n+7)}$ for the n^{th} or last term.

273 Analogous series may be reduced to the types in (268) and (270), or else the terms may be decomposed in the manner shewn in (272).

Ex.: $$\frac{1}{1.2.3}+\frac{4}{2.3.4}+\frac{7}{3.4.5}+\frac{10}{4.5.6}+\ldots\ldots$$

has for its general term

$$\frac{3n-2}{n(n+1)(n+2)}=-\frac{1}{n}+\frac{5}{n+1}-\frac{4}{n+2} \text{ by (235)},$$

and we may proceed as in (272) to find the sum of n terms.

The method of (272) includes the method known as "Summation by Subtraction," but it has the advantage of being more general and easier of application to complex series.

COMPOSITE FACTORIAL SERIES.

274 If the two series

$$(1-x)^{-5}=1+5x+\frac{5.6}{1.2}x^2+\frac{5.6.7}{1.2.3}x^3+\frac{5.6.7.8}{1.2.3.4}x^4+\ldots$$

$$(1-x)^{-3}=1+3x+\frac{3.4}{1.2}x^2+\frac{3.4.5}{1.2.3}x^3+\frac{3.4.5.6}{1.2.3.4}x^4+\ldots$$

be multiplied together, and the coefficient of x^4 in the product be equated to the coefficient of x^4 in the expansion of $(1-x)^{-8}$, we obtain as the result the sum of the composite series

$$5.6.7.8\times1.2+4.5.6.7\times2.3+3.4.5.6\times3.4$$
$$+2.3.4.5\times4.5+1.2.3.4\times5.6=\frac{4!\,2.11!}{7!\,4!}.$$

275 Generally, if the given series be

$$P_1Q_1+P_2Q_2+\ldots+P_{n-1}Q_{n-1} \quad\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where $\qquad Q_r=r(r+1)(r+2)\ldots(r+q-1),$

and $\qquad P_r=(n-r)(n-r+1)\ldots(n-r+p-1);$

the sum of $n-1$ terms will be

$$\frac{p!\,q!}{(p+q+1)!}\cdot\frac{(n+p+q-1)!}{(n-2)!}.$$

MISCELLANEOUS SERIES.

276 *Sum of the powers of the terms of an Arithmetical Progression.*

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} = S_1$$

$$1+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = S_2$$

$$1+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = S_3$$

$$1+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = S_4.$$

[By the method of Indeterminate Coefficients (234).

A general formula for the sum of the r^{th} powers of $1.2.3\ldots n$, obtained in the same way is

$$S_r = \frac{1}{r+1}n^{r+1}+\tfrac{1}{2}n^r+A_1n^{r-1}+\ldots A_{r-1}n,$$

where A_1, A_2, &c., are determined by putting $p=1, 2, 3,$ &c. successively in the equation

$$\frac{1}{2(p+1)!}$$

$$=\frac{1}{(p+2)!}+\frac{A_1}{r(p)!}+\frac{A_2}{r(r-1)(p-1)!}+\ldots\frac{A_p}{r(r-1)\ldots(r-p+1)}.$$

277 $$a^m+(a+d)^m+(a+2d)^m+\ldots+(a+nd)^m$$

$$=(n+1)a^m+S_1ma^{m-1}d+S_2C(m,2)a^{m-2}d^2$$
$$+S_3C(m,3)a^{m-3}d^3+\&c.$$

PROOF.—By Binomial Theorem and (276).

278 *Summation of a series partly Arithmetical and partly Geometrical.*

EXAMPLE.—To find the sum of the series $1+3x+5x^2+$ to n terms.

Let $\quad s = 1+3x+5x^2+7x^3+\ldots+(2n-1)\,x^{n-1},$

$\quad sx = \quad x+3x^2+5x^3+\ldots+(2n-3)\,x^{n-1}+(2n-1)\,x^n,$

$\therefore$ by subtraction,

$$s\,(1-x) = 1+2x+2x^2+2x^3+\ldots+2x^{n-1}-(2n-1)\,x^n$$

$$= 1+2x\,\frac{1-x^{n-1}}{1-x}-(2n-1)\,x^n,$$

$$\therefore\quad s = \frac{1-(2n-1)\,x^n}{1-x}+\frac{2x\,(1-x^{n-1})}{(1-x)^2}.$$

---

279 A general formula for the sum of n terms of

$$a+(a+d)\,r+(a+2d)\,r^2+(a+3d)\,r^3+\&\text{c.}$$

is
$$S = \frac{a-(a+\overline{n-1}\,d)\,r^n}{1-r}+\frac{dr\,(1-r^{n-1})}{(1-r)^2}.$$

Obtained as in (278).

Rule.—*Multiply by the ratio and subtract the resulting series.*

---

280 $$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots+x^{n-1}+\frac{x^n}{1-x}.$$

281 $$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots$$
$$\ldots+nx^{n-1}+\frac{(n+1)\,x^n-nx^{n+1}}{(1-x)^2}.$$

---

282 $$(n-1)\,x+(n-2)\,x^2+(n-3)\,x^3+\ldots+2x^{n-2}+x^{n-1}$$
$$= \frac{(n-1)\,x-nx^2+x^{n+1}}{(1-x)^2}.$$
By (253).

---

283 $$1+n+\frac{n\,(n-1)}{2\,!}+\frac{n\,(n-1)\,(n-2)}{3\,!}+\&\text{c.} = 2^n,$$

$$1-n+\frac{n\,(n-1)}{2\,!}-\frac{n\,(n-1)\,(n-2)}{3\,!}+\&\text{c.} = 0.$$

By making ~~$a=b$~~ $x=1$ in (125). errata

284 The series

$$1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{3\,!}-\frac{(n-5)(n-6)(n-7)}{4\,!}+\dots$$

$$\dots+(-1)^{r-1}\frac{(n-r-1)(n-r-2)\dots(n-2r+1)}{r\,!}$$

consists of $\frac{n}{2}$ or $\frac{n-1}{2}$ terms, and the sum is given by

$$S=\quad\frac{3}{n}\text{ if } n \text{ be of the form } 6m+3,$$

$$S=\quad 0 \text{ if } n \text{ be of the form } 6m\pm 1,$$

$$S=-\frac{1}{n}\text{ if } n \text{ be of the form } 6m,$$

$$S=\quad\frac{2}{n}\text{ if } n \text{ be of the form } 6m\pm 2.$$

Proof.—By (545), putting $p=x+y$, $q=xy$, and applying (546).

285 The series $n^r-n\,(n-1)^r+\frac{n\,(n-1)}{2\,!}(n-2)^r$

$$-\frac{n\,(n-1)(n-2)}{3\,!}(n-3)^r+\&\text{c.}\dots$$

takes the values 0, $n\,!$, $\tfrac{1}{2}n\,(n+1)\,!$

according as r is $<n$, $=n$, or $=n+1$.

Proof.—By expanding $(e^x-1)^n$, in two ways: first, by the Exponential Theorem and Multinomial; secondly, by the Bin. Th., and each term of the expansion by the Exponential. Equate the coefficients of x^r in the two results.

Other results are obtained by putting $r=n+2$, $n+3$, &c.

The series (285), when divided by $r\,!$, is, in fact, equal to the coefficient of x^r in the expansion of

$$\left\{x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\dots\right\}^n.$$

286 By exactly the same process we may deduce from the function $\{e^x - e^{-x}\}^n$ the result that the series

$$n^r - n(n-2)^r + \frac{n(n-1)}{2!}(n-4)^r - \&c.$$

takes the values 0 or $2^n . n!$, according as r is $< n$ or $= n$; this series, divided by $r!$, being equal to the coefficient of x^r in the expansion of

$$2^n \left\{ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right\}^n.$$

POLYGONAL NUMBERS.

287 The n^{th} term of the r^{th} order of polygonal numbers is equal to the sum of n terms of an Arith. Prog. whose first term is unity and common difference $r-2$; that is

$$= \frac{n}{2}\{2 + (n-1)(r-2)\} = n + \tfrac{1}{2}n(n-1)(r-2).$$

288 The sum of n terms

$$= \frac{n(n+1)}{2} + \frac{n(n-1)(n+1)(r-2)}{6}.$$

By resolving into two series.

Order.	n^{th} term.							
1	1	1	1	1	1	1	1	1
2	n	1	2	3	4	5	6	7
3	$\frac{1}{2}n(n+1)$	1	3	6	10	15	21	28
4	n^2	1	4	9	16	25	36	49
5	$\frac{1}{2}n(3n-1)$	1	5	12	22	35	51	70
6	$(2n-1)n$	1	6	15	28	45	66	91
...	... ...	...	...	...	...	...	...	...
r	$n + \frac{n(n-1)}{2}(r-2)$	1,	r,	$3+3(r-2)$,	$4+6(r-2)$,	$5+10(r-2)$,	$6+15(r-2)$,	&c.

In practice—to form, for instance, the 6^{th} order of polygonal numbers—write the first three terms by the formula, and form the rest by the method of differences.

Ex.:	1	6	15	28	45	66	91	120 ...	
	5	9	13	17	21	25	29 ...		
$[r-2=4]$	4	4	4	4	4	4 ...			

FIGURATE NUMBERS.

289 The n^{th} term of any order is the sum of n terms of the preceding order.

The n^{th} term of the r^{th} order is

$$\frac{n(n+1)\dots(n+r-2)}{(r-1)!} = H(n, r-1). \qquad \text{[By 98.}$$

290 The sum of n terms is

$$\frac{n(n+1)\dots(n+r-1)}{r!} = H(n, r).$$

Order.	Figurate Numbers.	n^{th} term.
1	1, 1, 1, 1, 1, 1	1
2	1, 2, 3, 4, 5, 6	n
3	1, 3, 6, 10, 15, 21	$\frac{n(n+1)}{1.2}$
4	1, 4, 10, 20, 35, 56	$\frac{n(n+1)(n+2)}{1.2.3}$
5	1, 5, 15, 35, 70, 126	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$
6	1, 6, 21, 56, 126, 252	$\frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$

HYPERGEOMETRICAL SERIES.

291 $$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\&c.$$

is convergent if x is <1,
and divergent if x is >1; (239 ii.)

and if $x=1$, the series is

convergent if $\gamma-\alpha-\beta$ is positive,
divergent if $\gamma-\alpha-\beta$ is negative, (239 iv.)
and divergent if $\gamma-\alpha-\beta$ is zero. (239 v.)

Let the hypergeometrical series (291) be denoted by $F(\alpha, \beta, \gamma)$; then, the series being convergent, it is shewn by induction that

292 $$\frac{F(\alpha,\beta+1,\gamma+1)}{F(\alpha,\beta,\gamma)}=\cfrac{1}{1-\cfrac{k_1}{1-\cfrac{k_2}{1-\&c.\ldots}}}$$ concluding with $$\cfrac{1-k_{2r-1}}{1-k_{2r}z_{2r}}$$

where k_1, k_2, k_3, &c., with z_{2r}, are given by the formulæ

$$k_{2r-1}=\frac{(\alpha+r-1)(\gamma+r-1-\beta)x}{(\gamma+2r-2)(\gamma+2r-1)}$$

$$k_{2r}=\frac{(\beta+r)(\gamma+r-\alpha)x}{(\gamma+2r-1)(\gamma+2r)}$$

$$z_{2r}=\frac{F(\alpha+r,\beta+r+1,\gamma+2r+1)}{F(\alpha+r,\beta+r,\gamma+2r)}.$$

The continued fraction may be concluded at any point with $k_{2r}z_{2r}$. When r is infinite, $z_{2r}=1$ and the continued fraction is infinite.

293 Let

$$f(\gamma) \equiv 1 + \frac{x^2}{1.\gamma} + \frac{x^4}{1.2.\gamma(\gamma+1)} + \frac{x^6}{1.2.3.\gamma(\gamma+1)(\gamma+2)} + \&c.$$

the result of substituting $\frac{x^2}{\alpha\beta}$ for x in (291), and making $\beta = \alpha = \infty$. Then, by last, or independently by induction,

$$\frac{f(\gamma+1)}{f(\gamma)} = \frac{1}{1+}\,\frac{p_1}{1+}\,\frac{p_2}{1+}\dots+\frac{p_m}{1+}\&c.$$

$$\text{with } p_m = \frac{x^2}{(\gamma+m-1)(\gamma+m)}.$$

294 In this result put $\gamma = \frac{1}{2}$ and $\frac{y}{2}$ for x, and we obtain by Exp. Th. (150),

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{y}{1+}\,\frac{y^2}{3+}\,\frac{y^2}{5+}\&c. \quad \text{the } r^{\text{th}} \text{ component being } \frac{y^2}{2r-1}.$$

Or the continued fraction may be formed by ordinary division of one series by the other.

295 $e^{\frac{m}{n}}$ is incommensurable, m and n being integers. From the last and (174), by putting $x = \frac{m}{n}$.

INTEREST.

If r be the Interest on £1 for 1 year,
n the number of years,
P the Principal,
A the amount in n years. Then

296 At Simple Interest $A = P(1+nr)$.

297 At Compound Interest $A = P(1+r)^n$. By (84).

298 But if the payments of Interest be made q times a year ... ... $\quad A = P\left(1+\dfrac{r}{q}\right)^{nq}.$

If A be an amount due in n years' time, and P the present worth of A. Then

299 At Simple Interest $\quad P = \dfrac{A}{1+nr}.$ By (296).

300 At Compound Interest $\quad P = \dfrac{A}{(1+r)^n}.$ By (297).

301 $\quad$ Discount $= A - P$.

ANNUITIES.

302 The amount of an Annuity of £1 in n years, at Simple Interest ... $\quad = n + \dfrac{n(n-1)}{2}r.$ By (82).

303 Present value of same $\quad = \dfrac{n+\frac{1}{2}n(n-1)r}{1+nr}.$ By (299).

304 Amount at Compound Interest ... ... $\quad = \dfrac{(1+r)^n-1}{(1+r)-1}.$ By (85).

Present worth of same $\quad = \dfrac{1-(1+r)^{-n}}{(1+r)-1}.$ By (300).

305 Amount when the payments of Interest are made q times per annum ... ... $\quad = \dfrac{\left(1+\dfrac{r}{q}\right)^{nq}-1}{\left(1+\dfrac{r}{q}\right)^{q}-1}.$ By (298).

Present value of same $\quad = \dfrac{1-\left(1+\dfrac{r}{q}\right)^{-nq}}{\left(1+\dfrac{r}{q}\right)^{q}-1}.$

306 Amount when the payments of the Annuity are made m times per annum ... ... $= \dfrac{(1+r)^n-1}{m\left\{(1+r)^{\frac{1}{m}}-1\right\}}$.

Present value of same $= \dfrac{1-(1+r)^{-n}}{m\left\{(1+r)^{\frac{1}{m}}-1\right\}}$.

307 Amount when the Interest is paid q times and the Annuity m times per annum ... $= \dfrac{\left(1+\frac{r}{q}\right)^{nq}-1}{m\left\{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1\right\}}$.

Present value of same $= \dfrac{1-\left(1+\frac{r}{q}\right)^{-nq}}{m\left\{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1\right\}}$.

PROBABILITIES.

309 If all the ways in which an event can happen be m in number, all being equally likely to occur, and if in n of these m ways the event would happen under certain restrictive conditions; then the probability of the restricted event happening is equal to $n \div m$.

Thus, if the letters of the alphabet be chosen at random, any letter being equally likely to be taken, the probability of a vowel being selected is equal to $\frac{5}{26}$. The number of unrestricted cases here is 26, and the number of restricted ones 5.

310 If, however, all the m events are not equally probable, they may be divided into groups of equally probable cases. The probability of the restricted event happening in each group separately must be calculated, and the sum of these probabilities will be the total probability of the restricted event happening at all.

Example.—There are three bags A, B, and C.

A contains 2 white and 3 black balls.
B ,, 3 ,, 4 ,,
C ,, 4 ,, 5 ,,

A bag is taken at random and a ball drawn from it. Required the probability of the ball being white.

Here the probability of the bag A being chosen $= \frac{1}{3}$, and the subsequent probability of a white ball being drawn $= \frac{2}{5}$.

Therefore the probability of a white ball being drawn from A

$$= \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}.$$

Similarly the probability of a white ball being drawn from B

$$= \frac{1}{3} \times \frac{3}{7} = \frac{1}{7}$$

And the probability of a white ball being drawn from C

$$= \frac{1}{3} \times \frac{4}{9} = \frac{4}{27}.$$

Therefore the total probability of a white ball being drawn

$$= \frac{2}{15} + \frac{1}{7} + \frac{4}{27} = \frac{401}{945}.$$

If a be the number of ways in which an event can happen, and b the number of ways in which it can fail, then the

311 Probability of the event happening $= \dfrac{a}{a+b}$.

312 Probability of the event failing $= \dfrac{b}{a+b}$.

Thus Certainty $= 1$.

If p, p' be the respective probabilities of two *independent* events, then

313 Probability of both happening $= pp'$.

314 ,, of not *both* happening $= 1-pp'$.

315 ,, of one happening and one failing

$$= p+p'-2pp'.$$

316 ,, of both failing $= (1-p)(1-p')$.

If the probability of an event happening in one trial be p, and the probability of its failing q, then

317 Probability of the event happening r times in n trials

$$= C(n, r)\, p^r q^{n-r}.$$

318 Probability of the event failing r times in n trials

$$= C(n, r)\, p^{n-r} q^r.$$ [By induction.

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319 Probability of the event happening *at least* r times in n trials = the sum of the *first* $n-r+1$ terms in the expansion of $(p+q)^n$.

320 Probability of the event failing *at least* r times in n trials = the sum of the *last* $n-r+1$ terms in the same expansion.

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321 The number of trials in which the probability of the same event happening amounts to p'

$$= \frac{\log(1-p')}{\log(1-p)}.$$

From the equation $(1-p)^x = 1-p'$.

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322 Definition.—When a sum of money is to be received if a certain event happens, that sum multiplied into the probability of the event is termed the expectation.

Example.—If three coins be taken at random from a bag containing one sovereign, four half-crowns, and five shillings, the expectation will be the sum of the expectations founded upon each way of drawing three coins. But this is also equal to the average value of three coins out of the ten; that is, $\frac{3}{10}$ths of 35 shillings, or 10*s.* 6*d.*

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323 The probability that, after r chance selections of the numbers $0, 1, 2, 3 \ldots n$, the sum of the numbers drawn will be s, is equal to the coefficient of x^s in the expansion of

$$(x^0 + x^1 + x^2 + \ldots + x^n)^r \div (n+1)^r.$$

324 The probability of the existence of a certain cause of an observed event out of several known causes, one of which *must* have produced the event, is proportional to the *a priori* probability of the cause existing multiplied by the probability of the event happening from it if it does exist.

Thus, if the *a priori* probabilities of the causes be P_1, P_2 ... &c., and the corresponding probabilities of the event happening from those causes Q_1, Q_2 ... &c., then the probability of the r^{th} cause having produced the event is

$$\frac{P_r Q_r}{\Sigma(PQ)}.$$

325 If P_1', P_2' ... &c. be the *a priori* probabilities of a second event happening from the same causes respectively, then, *after* the first event has happened, the probability of the second happening is $\dfrac{\Sigma(PQP')}{\Sigma(PQ)}$.

For this is the sum of such probabilities as $\dfrac{P_r Q_r P_r'}{\Sigma(PQ)}$, which is the probability of the r^{th} cause existing multiplied by the probability of the second event happening from it.

Ex. 1.—Suppose there are

4 vases containing each 5 white and 6 black balls,
2 vases containing each 3 white and 5 black balls,
and 1 vase containing 2 white and 1 black ball.

A white ball has been drawn, and the probability that it came from the group of 2 vases is required.

Here $P_1 = \frac{4}{7}$, $P_2 = \frac{2}{7}$, $P_3 = \frac{1}{7}$

$Q_1 = \frac{5}{11}$, $Q_2 = \frac{3}{8}$, $Q_3 = \frac{2}{3}$.

Therefore, by (324), the probability required is

$$\frac{\frac{2.3}{7.8}}{\frac{4.5}{7.11}+\frac{2.3}{7.8}+\frac{1.2}{7.3}} = \frac{99}{427}.$$

Ex. 2.—After the white ball has been drawn and replaced, a ball is drawn again; required the probability of the ball being black.

Here $\qquad P_1' = \frac{6}{11}, \qquad P_2' = \frac{5}{8}, \qquad P_3' = \frac{1}{3}.$

The probability, by (325), will be

$$\frac{\frac{4.5.6}{7.11.11}+\frac{2.3.5}{7.8.8}+\frac{1.2.1}{7.3.3}}{\frac{4.5}{7.11}+\frac{2.3}{7.8}+\frac{1.2}{7.3}} = \frac{58639}{112728}.$$

If the probability of the second ball being white is required, $Q_1Q_2Q_3$ must be employed instead of $P_1'P_2'P_3'$.

---

326 The probability of one event *at least* happening out of a number of events whose respective probabilities are a, b, c, &c., is

$$P_1 - P_2 + P_3 - P_4 + \&c.$$

where $\quad P_1$ is the probability of 1 event happening,
$\qquad\quad P_2 \quad$,, $\quad$,, $\quad$ 2 $\quad$,,

and so on. For, by (316), the probability is

$$1-(1-a)(1-b)(1-c) \ldots = \Sigma a - \Sigma ab + \Sigma abc -, \ldots$$

---

327 The probability of the occurrence of r assigned events and no more out of n events is

$$Q_r - Q_{r+1} + Q_{r+2} - Q_{r+3} + \&c.,$$

where Q_r is the probability of the r assigned events; Q_{r+1} the probability of $r+1$ events including the r assigned events.

For if a, b, c ... be the probabilities of the r events, and a', b', c' ... the probabilities of the excluded events, the required probability will be

$$abc \ldots (1-a')(1-b')(1-c') \ldots$$
$$= abc \ldots (1 - \Sigma a' + \Sigma a'b' - \Sigma a'b'c' + \ldots).$$

---

328 The probability of *any* r events happening and no more is

$$\Sigma Q_r - \Sigma Q_{r+1} + \Sigma Q_{r+2} - \&c.$$

Note.—If $a = b = c = \&c.$, then $\Sigma Q_r = C(n, r)\, Q_r$, &c.

INEQUALITIES.

330 $\dfrac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ lies between the greatest and least of the fractions $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \ldots \dfrac{a_n}{b_n}$, the denominators being all of the same sign.

PROOF.—Let k be the greatest of the fractions, and $\dfrac{a_r}{b_r}$ any other; then $a_r < kb_r$. Substitute in this way for each a. Similarly if k be the least fraction.

331 $$\frac{a+b}{2} > \sqrt{ab}.$$

332 $$\frac{a_1+a_2+\ldots+a_n}{n} > \sqrt[n]{a_1 a_2 \ldots a_n};$$

or, Arithmetic mean > Geometric mean.

PROOF.—Substitute both for the greatest and least factors their Arithmetic mean. The product is thus *increased* in value. Repeat the process indefinitely. The limiting value of the G. M. is the A. M. of the quantities.

333 $$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m,$$

excepting when m is a positive proper fraction.

PROOF: $$a^m+b^m = \left(\frac{a+b}{2}\right)^m \{(1+x)^m+(1-x)^m\},$$

where $x = \dfrac{a-b}{a+b}$. Employ Bin. Th.

334 $$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n} > \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m,$$

excepting when m is a positive proper fraction.

Otherwise.—*The Arithmetic mean of the m^{th} powers is greater than the m^{th} power of the Arithmetic mean, excepting when m is a positive proper fraction.*

Proof.—Similar to (332). Substitute for the greatest and least on the left side, employing (333).

336 If x and m are positive, and x and mx less than unity; then $$(1+x)^{-m} > 1-mx. \qquad (125, 240)$$

337 If x, m, and n are positive, and n greater than m; then, by taking x small enough, we can make

$$1+nx > (1+x)^m.$$

For x may be diminished until $1+nx$ is $> (1-mx)^{-1}$, and this is $> (1+x)^m$, by *last*.

338 If x be positive, $\log(1+x) < x$. (150)

If x be positive and > 1, $\log(1+x) > x - \frac{x^2}{2}$. (155, 240)

If x be positive and < 1, $\log\frac{1}{1-x} > x$. (156)

339 When n becomes infinite in the two expressions

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \quad\text{and}\quad \frac{3.5.7\ldots(2n+1)}{2.4.6\ldots 2n},$$

the first vanishes, the second becomes infinite, and their product lies between $\frac{1}{2}$ and 1.

Shewn by adding 1 to each factor (see 73), and multiplying the result by the original fraction.

340 If m be $> n$, and $n > a$,

$$\left(\frac{m+a}{m-a}\right)^m \text{ is } < \left(\frac{n+a}{n-a}\right)^n.$$